A.N. Tychonov, A.A. Samarski

Partial Differential Equations of Mathematical Physics

Volume 2

PARTIAL DIFFERENTIAL EQUATIONS OF MATHEMATICAL PHYSICS

Volume II

HOLDEN-DAY SERIES IN MATHEMATICAL PHYSICS

JULIUS J. BRANDSTATTER, *Editor*

A. N. Tychonov and A. A. Samarski

PARTIAL DIFFERENTIAL EQUATIONS OF MATHEMATICAL PHYSICS

Volume II

Translated by S. B. Radding

HOLDEN-DAY, INC.
San Francisco, Cambridge, London, Amsterdam

© Copyright 1967 by Holden-Day, Inc., 500 Sansome Street, San Francisco, California. All right reserved. No part of this book may be reproduced in any form, by mimeograph or any other means, without permission in writing from the publisher.

Library of Congress Catalog Card Number: 64-15462
Printed in the United States of America

EDITOR'S PREFACE

The present volume covers the three-dimensional aspects of the material contained in the first volume. There is an additional chapter on the special functions that arise in the solution of the problems treated; the basic properties and representations of these functions are derived in a simple and straightforward manner. As in the first volume, the topics in each chapter are motivated by their physical origins and are supplemented by examples, as well as by exercises. A few additional but pertinent references have also been included.

CONTENTS

5. SPATIAL WAVE PROPAGATION 381

5-1. Initial Value Problems 381
1. The mean value method 381
2. The method of descent 383
3. Physical interpretation 385
4. The method of images 387

5-2. The Kirchhoff Formula 388
1. Derivation of the Kirchhoff formula 388
2. Consequences of the Kirchhoff wave formula 392

5-3. Vibrations of a Bounded Spatial Region 394
1. General scheme for the method of separation of variables. Standing waves 394
2. Vibrations of a rectangular membrane 399
3. Vibrations of a circular membrane 402

Problems 407

5-4. Applications of Chapter 5 408
1. Reduction of the equations of the theory of elasticity to wave equations 408
2. The equations of the electromagnetic field 410
1. The equations of electromagnetic fields and boundary conditions 410
2. Potential of an electromagnetic field 414
3. The electromagnetic field of an oscillator 416

6. SPATIAL HEAT PROPAGATION 422

6-1. Heat Propagation in an Unbounded Space 422
1. Green's function 422
2. Heat propagation in an unbounded space 426

6-2. Heat Propagation in Bounded Regions 429
1. Scheme of the method of separation of variables 429
2. Cooling of a circular cylinder 432
3. Determination of the critical dimensions 434

6-3. Boundary-Valve Problems for Regions With Variable Boundaries 436
1. Green's formula for the heat-conduction equation and Green's function 436
2. Solution of the boundary-value problem 441

3. Green's function for an interval 443
6-4. Heat Potentials 445
1. Properties of heat potentials of single and double layers 445
2. Solution of boundary-value problems 447
Problems 449
6-5. Applications of Chapter 6 450
1. Diffusion of a cloud 450
2. The demagnetization of cylinders 453
3. The difference method for the heat-conduction equation 457

7. ELLIPTIC DIFFERENTIAL EQUATIONS (Continued) 465

7-1. Some Fundamental Problems which Lead to the Differential Equation $\Delta v+cv=0$ 465
1. Forced vibrations 465
2. Diffusion of gases with decomposition phenomena and with chain reactions 466
3. Diffusion in a moving medium 466
4. Formulation of the interior boundary-value problem for the equation $\Delta v+cv=0$ 467
7-2. Green's Function 468
1. Green's function 468
2. Integral representation of the solution 470
3. Potentials 473
7-3. Problems for an Unbounded Region: Radiation Principle 476
1. The equation $\Delta v+cv=-f$ in an unbounded space 476
2. The principle of limiting absorption 477
3. The principle of limiting amplitude 478
4. Radiation conditions 480
7-4. Problems of the Mathematical Theory of Diffraction 484
1. Statement of the problem 484
2. Uniqueness of the solutions 485
3. Diffraction by a sphere 488
Problems 495
7-5. Applications of Chapter 7 497
1. Waves in conducting tubes (wave guides) 497
2. Electromagnetic vibrations in cavity resonators 507
1. The eigenvibrations of cylindrical endovibrators 507
2. Electromagnetic energy of eigenvibrations 511
3. Generation of vibrations in an endovibrator 513
3. Skin effect 515
4. The propagation of radio waves on the surface of the earth 519

APPENDIX: SPECIAL FUNCTIONS 525

A-1 Introduction 525
1. Differential equations of special functions 525

2. Formulation of the boundary value problems in the case $k(a)=0$ 526
A-2 Cylindrical Functions 532
1. The cylindrical functions 532
1. Power series 533
2. Recursion formulas 538
3. The Bessel function of the first kind of order $n+1/2$ 539
4. Asymptotic representation of cylindrical functions for large x 540
2. Boundary-value problems for the Bessel differential equation 542
3. The different types of cylindrical functions 545
1. Hankel functions 545
2. Hankel and Neumann functions 546
3. Bessel functions with imaginary arguments 548
4. The function $K_0(x)$ 549
4. Integral representations and asymptotic representations of the Bessel functions 553
1. Integral representations for the Bessel functions of integral order 553
2. Asymptotic representations of Bessel functions of the first kind 556
5. The Fourier-Bessel integral and some integrals which contain Bessel functions 559
1. Fourier-Bessel integral 559
2. Some integrals whose integrands contain Bessel functions 559
6. Representation of cylindrical functions by contour integrals 563
1. Representation of cylindrical functions by contour integrals 563
2. The saddle point method. Asymptotic representations 568
A-3 Spherical Functions 571
1. Legendre polynomials 571
1. The generating function and Legendre polynomials 571
2. A recursion formula 573
3. The Legendre differential equation 573
4. The orthogonality of the Legendre polynomials 574
5. The norm of the Legendre polynomials 576
6. A differential relation for the Legendre polynomials 577
7. An integral formula. The boundedness of the Legendre polynomials 578
8. The associated Legendre polynomials 580
9. The closedness of systems of associated Legendre polynomials 582
2. Harmonic polynomials and spherical functions 584
1. Harmonic polynomials 584
2. Spherical functions 585

3. The orthogonality of the system of spherical functions 588
4. The completeness of systems of spherical functions 590
5. The expansion in spherical functions 591
3. Some examples of the application of spherical functions 594
1. The polarization of a sphere in a homogeneous field 594
2. The eigenvibrations of a sphere 597
3. The exterior boundary-value problem for the sphere 599
A-4 The Tschebyscheff-Hermite and the Tschebyscheff-Laguerre Polynomials 601
1. The Tschebyscheff-Hermite polynomial 601
2. The Tschebyscheff-Laguerre polynomials 604
3. Simple problems for the Schrödinger equation 610
1. The Schrödinger equation 610
2. The harmonic oscillator 611
3. The eigenvalue problem of rotators 613
4. Motion of electrons in a Coulomb field 614

Index 619

PARTIAL DIFFERENTIAL EQUATIONS OF MATHEMATICAL PHYSICS

Volume II

5

SPATIAL WAVE PROPAGATION

5-1. INITIAL VALUE PROBLEM

1. The mean value method

In this paragraph we shall concern ourselves with the three-dimensional wave equation

$$\Delta u - \frac{1}{a^2}\frac{\partial^2 u}{\partial t^2} = -f \tag{5-1.1}$$

and determine the solution which satisfies the initial conditions

$$\begin{aligned} u(x, y, z, 0) &= \varphi(x, y, z) , \\ u_t(x, y, z, 0) &= \psi(x, y, z) . \end{aligned} \tag{5-1.2}$$

This problem is of fundamental significance for the theory of sound propagation, for the investigation of the propagation of electromagnetic fields, as well as for a number of other fields of physics. For simplicity we shall first assume that $f = 0$.

To determine the function u at a point M_0 we introduce the new function

$$\bar{u}(r, t) = \frac{1}{4\pi r^2}\int_{S_r^{M_0}}\int u\, dS = \frac{1}{4\pi}\int_{S_r^{M_0}}\int u\, d\Omega . \tag{5-1.3}$$

Here $S_r^{M_0}$ denotes the spherical surface of radius r about $M_0(x_0, y_0, z_0)$, and $d\Omega = dS/r^2$ is the element of solid angle corresponding to dS. Obviously, $\bar{u}(r, t)$ is the average value of the function u on the spherical surface $S_r^{M_0}$ with radius r about M_0. We see at once that

$$u(M_0, t_0) = \bar{u}(0, t_0) .$$

We shall soon show that the function u so defined satisfies the homogeneous equation. For this pourpose we write the Laplace differential expression Δu in spherical coordinates

$$\Delta u = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial u}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 u}{\partial\varphi^2} .$$

Since $\bar{u}$ does not depend on θ and φ,

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\bar{u}}{\partial r}\right) - \frac{1}{a^2}\frac{\partial^2\bar{u}}{\partial t^2} = 0$$

must be satisfied, provided our assertion is correct. The integration of the homogeneous Eq. (5-1.1) (when $f = 0$) over a region $T_r^{M_0}$ bounded by the spherical surface $S_r^{M_0}$ gives

$$\frac{1}{a^2}\frac{\partial^2}{\partial t^2}\iiint_{T_r^{M_0}} u\, d\tau = \iiint_{T_r^{M_0}} \Delta u\, d\tau = \iint_{S_r^{M_0}} \frac{\partial u}{\partial n}\, dS . \tag{5-1.4}$$

If now we recall that the unit normal vector of $S_r^{M_0}$ is in the direction of the radius of the sphere and that the surface area element $dS = r^2\, d\Omega$, then we find

$$\iint_{S_r^{M_0}} \frac{\partial u}{\partial n}\, dS = \iint_{S_r^{M_0}} \frac{\partial u}{\partial r} r^2\, d\Omega = r^2 \frac{\partial}{\partial r}\left[\iint_{S_r^{M_0}} u\, d\Omega\right].$$

Therefore, if on the left side of Eq. (5-1.4) we set $d\tau = r^2\, dr\, d\Omega$, we obtain

$$\frac{1}{a^2}\frac{\partial^2}{\partial t^2}\int_0^r r'^2\, dr'\left[\iint_{S_{r'}^{M_0}} u\, d\Omega\right] = r^2 \frac{\partial}{\partial r}\left[\iint_{S_r^{M_0}} u\, d\Omega\right].$$

By differentiation of the last equation with respect to r and division by r^2, we conclude that $\bar{u}$ satisfies the equation

$$\frac{1}{a^2}\frac{\partial^2 \bar{u}}{\partial t^2} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial \bar{u}}{\partial r}\right).$$

Now, the Laplace differential expression has the property that it assumes the form

$$\Delta U = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial U}{\partial r}\right) = \frac{1}{r}\frac{\partial^2 (rU)}{\partial r^2}$$

for a function $U = U(r, t)$ which does not depend on φ and θ. By using this property of the Laplace operator, it is easy to see that for every spherically symmetric solution $\bar{u}$ of the three-dimensional wave equation, the substitution $v = r\bar{u}$ leads to the one-dimensional wave equation

$$\frac{\partial^2 v}{\partial r^2} = \frac{1}{a^2}\frac{\partial^2 v}{\partial t^2} .$$

The general solution of this equation has the form

$$v = r\bar{u} = f_1\left(t - \frac{r}{a}\right) + f_2\left(t + \frac{r}{a}\right).$$

For $r = 0$, because of the boundedness of $\bar{u}$, we obtain

$$0 = f_1(t) + f_2(t) \qquad \text{or} \qquad f_1(t) = -f_2(t) = f(t)$$

for arbitrary values of t; that is,

$$v = r\bar{u} = f\left(t + \frac{r}{a}\right) \quad f\left(t - \frac{r}{a}\right).$$

If we differentiate this formula with respect to r, then we obtain

$$u + r\frac{\partial \bar{u}}{\partial r} = \frac{1}{a}\left[f'\left(t - \frac{r}{a}\right) + f'\left(t + \frac{r}{a}\right)\right],$$

from which, for $r = 0$ and $t = t_0$, there follows

$$\bar{u}(0, t_0) = u(M_0, t_0) = \frac{2}{a} f'(t_0) .$$

For the solution of the initial value problem formulated above, the function $f(t)$ must now be expressed in terms of the functions φ and ψ occurring in the initial conditions. By addition of the equations

$$\frac{\partial}{\partial r}(r\bar{u}) = \frac{1}{a}\left[f'\left(t - \frac{r}{a}\right) + f'\left(t + \frac{r}{a}\right)\right],$$

$$\frac{1}{a}\frac{\partial}{\partial t}(r\bar{u}) = \frac{1}{a}\left[-f'\left(t - \frac{r}{a}\right) + f'\left(t + \frac{r}{a}\right)\right],$$

we obtain

$$\frac{\partial}{\partial r}(r\bar{u}) + \frac{1}{a}\frac{\partial}{\partial t}(r\bar{u}) = \frac{2}{a} f'\left(t + \frac{r}{a}\right).$$

For $t = 0$ and $r = at_0$, it follows that

$$u(M_0, t_0) = \left[\frac{\partial}{\partial r}(r\bar{u}) + \frac{1}{a}\frac{\partial}{\partial t}(r\bar{u})\right]_{r=at_0, t=0}$$

or, if we introduce the expression (5-1.3) for $\bar{u}$,

$$u(M_0, t_0) = \frac{1}{4\pi}\left[\frac{\partial}{\partial r}\int_{S_r^{M_0}}\!\!\int ru\, d\Omega + \frac{1}{a}\int_{S_r^{M_0}}\!\!\int r\frac{\partial u}{\partial t}\, d\Omega\right]_{r=at_0, t=0} .$$

Here, if we substitute the initial values and drop the index 0 from M_0 and t_0, we finally arrive at the so-called Poisson formula

$$u(M, t) = \frac{1}{4\pi a}\left[\frac{\partial}{\partial t}\int_{S_{at}^{M}}\!\!\int \frac{\varphi}{r}\, dS + \int_{S_{at}^{M}}\!\!\int \frac{\psi}{r}\, dS\right]. \tag{5-1.5}$$

This formula was derived under the assumption that a solution exists for the problem considered. Our formulation simultaneously shows the uniqueness of the solution for the prescribed initial conditions (see Section 2-2). As we can prove directly, the Poisson formula under very general assumptions on φ and ψ yields the solution of the problem.

2. The method of descent[1]

In the x, y, z-space, formula (5-1.5), which was obtained in the preceding section, gives the solution of the homogeneous wave equation with initial conditions which are, in general, arbitrary functions of the variables x, y, z. If the initial values φ and ψ do not depend on z, then obviously the function u defined by (5-1.5) is also indepedent of z. Consequently, this function satisfies the equation

$$u_{xx} + u_{yy} - \frac{1}{a^2} u_{tt} = 0 ,$$

[1] This designation is due to J. Hadamard; see R. Courant and D. Hilbert, *Methods of Mathematical Physics*, Vol. II, Interscience, 1962, pp. 205-206, 686-688.

and the initial conditions

$$u(x, y, 0) = \varphi(x, y) ,$$
$$u_t(x, y, 0) = \psi(x, y) .$$

The Poisson formula, therefore, leads not only to solutions of spatial problems but of plane problems as well.

In formula (5-1.5) the integration extends over the spherical surface S_{at}^M. Since the initial value is independent of z, the integration over the upper hemisphere can be replaced by an integration over the circular surface Σ_{at}^M, which we obtain by cutting the sphere S_{at}^M with the x, y-plane (Figure 77).

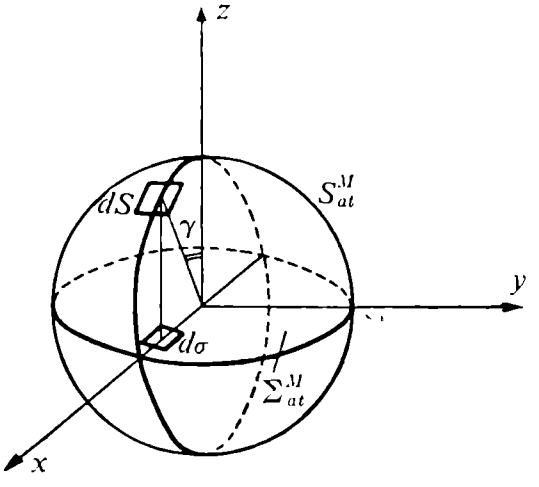


FIG. 77

The surface area element dS of the sphere is related to the surface element $d\sigma$ of the circle by

$$d\sigma = dS \cos\gamma .$$

Therefore, we find

$$\cos\gamma = \frac{\sqrt{(at)^2 - \rho^2}}{at} = \frac{\sqrt{(at)^2 - (x-\xi)^2 - (y-\eta)^2}}{at} .$$

We perform the integration over the lower hemisphere precisely as before so that the integral over the circle is taken twice. Therefore, we arrive at the formula

$$u(M, t) = u(x, y, t) = \frac{1}{2\pi a}\left[\frac{\partial}{\partial t}\int_{\Sigma_{at}^M}\int \frac{\varphi(\xi, \eta)\, d\xi\, d\eta}{\sqrt{(at)^2 - (x-\xi)^2 - (y-\eta)^2}} + \int_{\Sigma_{at}^M}\int \frac{\psi(\xi, \eta)\, d\xi\, d\eta}{\sqrt{(at)^2 - (x-\xi)^2 - (y-\eta)^2}}\right], \tag{5-1.6}$$

in which the integration is to be extended over the interior of the circle of radius at with center (x, y).

Completely analogous is the case where the initial values φ and ψ depend on the variable x alone. In this case, with the aid of formula (5-1.5) we find a function $u(x, t)$ which is the solution of the equation

$$u_{xx} - \frac{1}{a^2} u_{tt} = 0$$

with the initial conditions

$$u(x, 0) = \varphi(x) , \qquad u_t(x, 0) = \psi(x) .$$

For this purpose we introduce polar coordinates whose polar axis has the direction of the x-axis. The surface area element dS then assumes the form

$$dS = r^2 \sin\theta \, d\theta \, d\varphi = r \, d\varphi \, d\xi ,$$

since

$$\xi = x + r\cos\theta , \qquad d\xi = -r \sin\theta \, d\theta .$$

Hence, by integration of the Poisson formula (5-1.5) with respect to φ we obtain

$$u(x, t) = \frac{1}{2a}\left[\frac{\partial}{\partial t}\int_{x-at}^{x+at} \varphi(\xi)\, d\xi + \int_{x-at}^{x+at} \psi(\xi)\, d\xi\right] .$$

Further, if we carry out the differentiation with respect to t in the first integral, we arrive finally at d'Alembert's formula (see Section 2-2)

$$u(x, t) = \frac{\varphi(x + at) + \varphi(x - at)}{2} + \frac{1}{2a}\int_{x-at}^{x+at} \psi(\xi)\, d\xi . \tag{5-1.7}$$

The wave equation with three, two, or one-position variables is known, respectively, as the spherical, cylindrical, and plane wave equation. This terminology corresponds completely to the method of descent used above because for the solution of the wave equation in the two-dimensional or one-dimensional case, we started from the corresponding three-dimensional problem and thus "descended" to a smaller number of variables. The solutions obtained for two variables or one variable correspond to cylindrical and plane waves, respectively.

The descent method is applicable not only to the wave equation, but also to other types of differential equations. In a series of cases, by starting with this method, we obtain from the solution of the corresponding equation in several variables, the solution of the equation with fewer number of variables.

3. Physical interpretation

Formulas (5 1.5) and (5-1.6) make it possible to obtain an idea of the physical form of the propagation of spherical and cylindrical waves. We shall consider first the three-dimensional case. As we have already seen, the physical character of the propagation processes for three variables can be distinguished essentially from that of two variables.[2]

We shall limit ourselves to the investigation of a local excitation, that is, we shall assume that the initial state described by the functions $\varphi > 0$, $\psi > 0$ is different from 0 only in a certain bounded region T_0. First, we consider

[2] Usually they are the position variables.

the change of state $u(M_0, t)$ at a point M_0, which could lie outside of the region T_0.

The state u at the point M_0 for time t is determined according to formula (5-1.5) by the initial state at those points which lie on the sphere $S_{at}^{M_0}$ of radius at with center M_0. The function $u(M_0, t)$ is then different from 0 only if the sphere $S_{at}^{M_0}$ intersects the region T_0 of initial values. Let d and D be the nearest and farthest distances of the point M_0 from the points of the region T_0 (Figure 78). If t is sufficiently small ($t < t_1 = d/a$), then $S_{at}^{M_0}$ does not cut

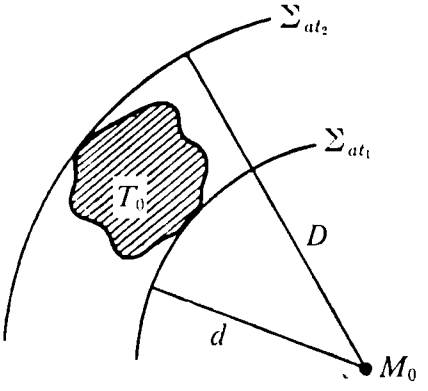


FIG. 78

the region T_0, and the surface integral in formula (5-1.5) is equal to zero; the excitation has not yet penetrated up to the point M_0. From time $t_1 = d/a$ up to time $t_2 = D/a$, the sphere $S_{at}^{M_0}$ ($t_1 < t < t_2$) cuts the region T_0; the surface integral in (5-1.5) in general is different from zero; the point M_0 is in an excited state. For larger values of t, the region T_0 lies in the interior of the sphere $S_{at}^{M_0}$, and the corresponding surface integral vanishes; the excitation penetrates beyond the point M_0. Therefore, in the case of propagation of local excitations in space, no secondary effects occur.

We now consider the instantaneous spatial form of the excitation $u(M, t_0)$ for time t_0. The points M which are found in the excited state are characterized by the fact that the initial excitation cuts the spheres $S_{at_0}^{M}$ of the region T_0. In other words, the set W of points on which the excitation is different from 0 consists of those points which lie on a sphere $S_{at_0}^{P}$ of radius at_0 about a center P of the region T_0. The envelopes of the family of spheres $S_{at_0}^{P}$ form the boundary of the region W. The outer envelope is called the forward front, the inner envelope the backward front of the propagating wave. Figure 79 shows the forward and backward fronts of the wave [(1) and (2)] for the case where the region T_0 is a sphere of radius R_0.

Accordingly, the initial excitation, which is spatially localized, produces a localized excitation at each point M_0 of space-time; hence there follows wave propagation with a sharply defined forward and backward front (Huygen's principle).

We turn now to the consideration of the two-dimensional case. Let the initial excitation be prescribed in a region S_0 of the x, y-plane. We are interested in the change of state $u(M_0, t)$ at a point M_0 which lies outside of

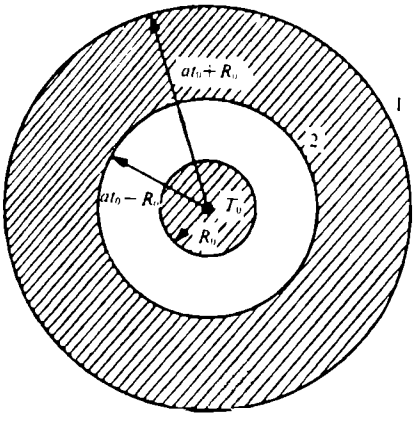


FIG. 79

S_0. On the basis of formula (5-1.6), the state $u(M_0, t)$ at M_0 for time t is determined by the initial value at those points P which belong to the circle $\Sigma_{at_0}^{M_0}$ with radius at_0 about M_0. For $t < t_1 = d/a$ (d denotes the shortest distance of the point M_0 from the points of region S_0), $u(M_0, t) = 0$; the excitation still has not penetrated up to the point M_0. For $t > t_1$, $u(M_0, t) \neq 0$. This means that at the time $t = t_1$, at the point M_0, an excitation occurs which, in general, increases and then, from a fixed time on, decreases to the value 0 (for $t \to \infty$).

The essential difference between the plane and the spatial case is in their secondary effects. The influence of the initial excitation localized in a bounded plane region is not localized temporally. In contrast such an influence is characterized by its longer lasting secondary effects. Therefore Huygen's principle is not valid in this case.

The instantaneous form of the excitation in the plane case consists of a precisely defined forward front, but no corresponding backward front. Problems in two variables can be treated as spatial problems in which the initial excitation is given on an infinitely long cylinder and does not depend on the third coordinate. In this way we can also easily obtain an idea of their secondary processes.

4. The method of images

The initial value problem for the wave equation can be solved by the method of images, provided the region under consideration is bounded by planes.

Thus we treat the following problem for the half space $z > 0$:

Determine the solution of the wave equation

$$Ju = \frac{1}{a^2} u_{tt}$$

which satisfies the initial conditions

$$\begin{aligned} u(x, y, z, 0) &= \varphi(x, y, z) , \\ u_t(x, y, z, 0) &= \psi(x, y, z) , \end{aligned} \qquad (z \geqq 0)$$

and the boundary conditions

$$u|_{z=0} = 0 \qquad \text{or} \qquad \left.\frac{\partial u}{\partial z}\right|_{z=0} = 0 .$$

The solution of this problem is given by formula (5-1.5) if the initial conditions are continued in the entire space; indeed, as an odd continuation (for $u|_{z=0} = 0$) we have

$$\varphi(x, y, z) = -\varphi(x, y, -z) , \qquad \psi(x, y, z) = -\psi(x, y, -z) ,$$

or as an even continuation (for $(\partial u/\partial z)|_{z=0} = 0$),

$$\varphi(x, y, z) = \varphi(x, y, -z) , \qquad \psi(x, y, z) = \psi(x, y, -z) .$$

We shall now prove that for odd continuations of the functions φ and ψ, the boundary condition $u|_{z=0} = 0$ is automatically satisfied. In fact,

$$u(P, t) = u(x, y, 0, t) = \frac{1}{4\pi a}\left[\frac{\partial}{\partial t}\int_{S_{at}^P}\int \frac{\varphi(\xi, \eta, \zeta)}{at}\, ds + \int_{S_{at}^P}\int \frac{\psi(\xi, \eta, \zeta)}{at}\, ds\right] = 0 ,$$

since the surface integral over the sphere, whose center lies at the points of the plane $z = 0$, is equal to zero for odd continuations of φ and ψ.

We treat analogously the problem of a stratified plane $0 \leqq z \leqq l$ with boundary conditions of the first or second kind and corresponding initial conditions

$$u = 0 \quad \text{for} \quad z = 0 \quad \text{and} \quad z = l ,$$

or

$$\frac{\partial u}{\partial z} = 0 \quad \text{for} \quad z = 0 \quad \text{and} \quad z = l .$$

Formula (5-1.5) immediately yields the solution, if the initial conditions are continued evenly (or oddly) with respect to planes $z = 0$ and $z = l$. The initial functions φ and ψ defined in this manner are then periodic functions of z with period $2l$.

If the initial functions φ and ψ in the layer $0 < z < l$ are chosen so that they differ from 0 only in a region T_0, then the continued functions differ from 0 only in a sequence of regions T_n. We obtain these from T_0 by images. The function $u(M, t)$ is then, for each M and t, a sum of finitely many summands, which are determined by the excitations in the regions T_n (see Section 2-2.5). The physical significance of this procedure lies in the fact that during a finite time interval, finitely many reflections from the walls $z = 0$ and $z = l$ occur. Correspondingly, we solve the initial value problem for a parallelepiped.

5-2. THE KIRCHHOFF FORMULA

1. Derivation of the Kirchhoff formula

To solve the equation of the vibrating string,

$$u_{xx} - \frac{1}{a^2} u_{tt} = f ,$$

by means of the wave propagation method, the idea of the characteristic angle is used extensively. Here, to solve the two- or three-dimensional wave equation

$$\Delta u - \frac{1}{a^2} u_{tt} = -f , \tag{5-2.1}$$

we use the surface

$$\frac{1}{a} r_{MM_0} = |t - t_0| ,$$

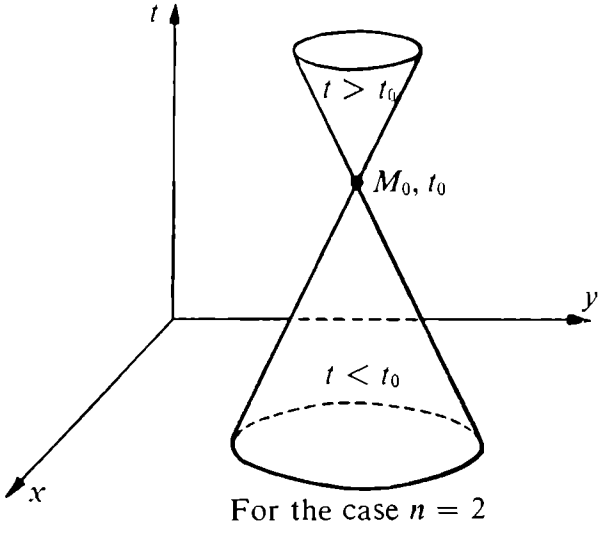
with the so-called characteristic cone occurring at point M_0 and time t_0. The totality of points of the phase space (M, t) at which a signal arrives, which is sent at the time t_0 from the point M_0 and which propagates with velocity a, is determined by the equation

$$\frac{1}{a} r_{MM_0} = t - t_0 , \qquad (t > t_0) ,$$

i.e., the points form the upper part of the characteristic cone belonging to the point M_0. Correspondingly, a signal sent at the time t from the point M reaches the point M_0 at time t_0, provided

$$\frac{1}{a} r_{MM_0} = t_0 - t , \qquad (t < t_0) .$$

The geometric locus of the points (M, t) is therefore the lower part of the characteristic cone (Figure 80).



For the case $n = 2$

FIG. 80

To determine a solution $u(M, t)$ of Eq. (5-2.1) at the point (M_0, t_0), we introduce in place of time t, a "local time" t^*, corresponding to the point M_0, by setting

$$t^* = t - \left(t_0 - \frac{r_{MM_0}}{a}\right) .$$

Now if we also introduce a polar coordinate system (r, θ, φ) attached to the

point M_0, we arrive at the new coordinate system

$$r^* = r\,, \qquad \theta^* = \theta\,, \qquad \varphi^* = \varphi\,, \qquad t^* = t - \left(t_0 - \frac{r}{a}\right).$$

At this point the question arises as to which equation the function

$$u(r, \theta, \varphi, t) = u\left(r^*, \theta^*, \varphi^*, t^* + t_0 - \frac{r}{a}\right) = U(r^*, \theta^*, \varphi^*, t^*)$$

must satisfy, if $u(M, t)$ satisfies Eq. (5-2.1). In polar coordinates the Laplace differential expression has the form

$$\Delta u = \frac{\partial^2 u}{\partial r^2} + \frac{2}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial u}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 u}{\partial \varphi^2}\,.$$

we note the relations

$$u_r = U_{r^*} + \frac{1}{a}U_{t^*}\,, \qquad u_\theta = U_{\theta^*}\,, \qquad u_{\theta\theta} = U_{\theta^*\theta^*}\,,$$

$$u_\varphi = U_{\varphi^*}\,, \qquad u_{\varphi\varphi} = U_{\varphi^*\varphi^*}\,,$$

$$u_{rr} = U_{r^*r^*} + \frac{2}{a}U_{r^*t^*} + \frac{1}{a^2}U_{t^*t^*}\,, \qquad u_t = U_{t^*}\,, \qquad u_{tt} = U_{t^*t^*}$$

between the derivatives with respect to the new and old coordinates. Eq. (5-2.1), therefore, is transformed into

$$\Delta U = -\frac{2}{ar^*}\frac{\partial}{\partial r^*}(r^* U_{t^*}) - F(r^*, \theta^*, \varphi^*, t^*)\,, \tag{5-2.2}$$

where

$$F(r^*, \theta^*, \varphi^*, t^*) = f(r, \theta, \varphi, t)\,.$$

Now let the point $M_0(x_0, y_0, z_0)$ lie in a spatial region T, which is bounded by the surface S. Then we interpret Eq. (5-2.2) as an inhomogeneous Laplace differential equation in which t^* is a parameter, and we apply Green's formula (Chapter 4) to the region T. Thus, if we set $t^* = 0$, then we obtain[3]

$$4\pi U(M_0, 0) = \int_S\int\left[\frac{1}{r^*}\frac{\partial U}{\partial n} - U\frac{\partial}{\partial n}\left(\frac{1}{r}\right)\right]dS + \iiint_T \frac{2}{ar^{*2}}\frac{\partial}{\partial r^*}\left(r^*\frac{\partial U}{\partial t^*}\right)d\tau + \iiint_T \frac{F}{r^*}\,d\tau\,.$$

The point M_0 here is the origin of the polar coordinate system. Therefore, the volume integral in the last expression is to be understood as follows: If T_ε is a sphere of radius ε about M_0, then the integral is extended over $T - T_\varepsilon$ and we pass to the limit as $\varepsilon \to 0$. For the second summand we obtain

$$I_\varepsilon = \iiint_{T-T_\varepsilon} \frac{2}{ar^{*2}}\frac{\partial}{\partial r^*}\left(r^*\frac{\partial U}{\partial t^*}\right)d\tau = \iiint_{T-T_\varepsilon} \frac{2}{a}\frac{\partial}{\partial t^*}\left(r^*\frac{\partial U}{\partial t^*}\right)\sin\theta^*\, dr^*\, d\theta^*\, d\varphi^*\,,$$

[3] According to the conventions in Chapter 4, the signs in the formula correspond to the exterior normals.

and by integration over r^*

$$I_\varepsilon = \int_S \int \frac{2}{ar^*} \frac{\partial U}{\partial t^*} \cos(n, r^*)\, dS - \int_{S_\varepsilon} \int \frac{2}{ar^*} \frac{\partial U}{\partial t^*}\, dS ,$$

since

$$dS_n = dS \cos(n, r^*) = r^{*2} \sin\theta^*\, d\theta^*\, d\varphi^* .$$

The second summand in I_ε approaches 0 as $\varepsilon \to 0$, since the area of the surface S_ε equals $4\pi\varepsilon^2$. Therefore, for the limit value of I_ε as $\varepsilon \to 0$, we find

$$I_0 = \iint_T \int \frac{2}{ar^{*2}} \frac{\partial}{\partial r^*}\left(r^* \frac{\partial U}{\partial t^*}\right) d\tau = \int_S \int \frac{2}{ar^*} \frac{\partial U}{\partial t^*} \frac{dr^*}{dn}\, dS ,$$

since[4]

$$\cos(n, r^*) = \frac{dr^*}{dn} .$$

Therefore,

$$4\pi U(M_0, 0) = \int_S \int \left(\frac{1}{r^*} \frac{\partial U}{\partial n} - U \frac{\partial}{\partial n}\left(\frac{1}{r^*}\right) + \frac{2}{ar^*} \frac{\partial U}{\partial t^*} \frac{dr^*}{dn}\right) dS + \iint_T \int \frac{F}{r^*}\, d\tau .$$

We now return to the original variables and to the function u

$$u(M, t) = U(M, t^*) , \qquad \left(t = t^* + t_0 - \frac{r_{M_0 M}}{a}\right) .$$

Hence there follows

$$U(M_0, 0) = u(M_0, t_0)$$

and

$$\frac{\partial u}{\partial n} = \frac{\partial U}{\partial n} + \frac{1}{a} \frac{\partial U}{\partial t^*} \frac{dr}{dn} .$$

Finally, we obtain for the function $u(M_0, t_0)$ the so-called Kirchhoff wave formula[5]

$$u(M_0, t_0) = \frac{1}{4\pi} \int_S \int \left\{\frac{1}{r}\left[\frac{\partial u}{\partial n}\right] - [u] \frac{\partial}{\partial n}\left(\frac{1}{r}\right) + \frac{1}{ar}\left[\frac{\partial u}{\partial t}\right] \frac{\partial r}{\partial n}\right\} dS_M + \frac{1}{4\pi} \iint_T \int \frac{[f]}{r}\, d\tau_M . \tag{5-2.3}$$

The square brackets signify that the corresponding function values for $t^* = 0$, that is for $t = (t_0 - r_{M_0 M})/a$, are to be taken, so that $[f] = f(M, t - r_{M_0 M}/a)$.

[4] For this transformation we use the expression $d\tau = r^{*2}\, d\Omega\, dr^{*2}$, integrate over r^* and, finally, set $d\Omega = dS/r^{*2}$.

[5] The Kirchhoff formula was generalized by S. L. Sobolev for the hyberbolic differential equation with an even number of variables. By means of this Kirchhoff-Sobolev formula, the solution of initial value problems for such equations is possible. See S. L. Sobolev, "On a Generalization of the Kirchhoff Formula," *Doklady*, New Series, 1933, p. 256.

2. Consequences of the Kirchhoff wave formula

Formula (5-2.3) is used for an entire series of problems. As a first example, we consider the following initial value problem:

Determine a solution of the wave equation

$$\Delta u - \frac{1}{a^2} u_{tt} = 0 ,$$

which is defined in the entire space and satisfies the initial conditions

$$u|_{t=0} = \varphi(x, y, z) , \qquad u_t|_{t=0} = \psi(x, y, z) .$$

The upper part of the characteristic cone $r = a(t_0 - t)$ of a point (M_0, t_0) cuts the manifold $t = 0$ in the sphere $S_{at_0}^{M_0}$ $(r = at_0)$ with a radius at_0 about M_0. Now we set $S = S_{at_0}^{M_0}$ and apply formula (5-2.3). For each function $v(M, t)$, $[v]$ on the sphere $S_{at_0}^{M_0}$ has the form

$$[v] = v\left(M, t_0 - \frac{r_{M_0 M}}{a}\right) = v(M, 0) .$$

Therefore

$$[u]_{S_{at_0}^{M_0}} = u(M, 0) = \varphi(M) ,$$

$$\left[\frac{\partial u}{\partial t}\right]_{S_{at_0}^{M_0}} = \frac{\partial u}{\partial t}(M, 0) = \psi(M) ,$$

$$\left[\frac{\partial u}{\partial n}\right]_{S_{at_0}^{M_0}} = \frac{\partial u}{\partial r}(M, 0) = \frac{\partial \varphi}{\partial r}(M) .$$

Furthermore, we find

$$\frac{1}{r}\left[\frac{\partial u}{\partial n}\right] - [u]\frac{\partial}{\partial n}\left(\frac{1}{r}\right) = \frac{1}{r}\frac{\partial \varphi}{\partial r} - \varphi\frac{\partial}{\partial r}\left(\frac{1}{r}\right) = \frac{1}{r^2}\frac{\partial}{\partial r}(r\varphi) .$$

If we insert this expression into formula (5-2.3), we obtain

$$\begin{aligned} u(M_0, t_0) &= \frac{1}{4\pi}\int_{S_{at_0}^{M_0}}\left\{\frac{1}{r^2}\frac{\partial}{\partial r}(r\varphi) + \frac{1}{ar}\psi\right\} dS \\ &= \frac{1}{4\pi}\left[\frac{\partial}{\partial r}\int_{S_{at_0}^{M_0}} r\varphi \, d\Omega + \frac{1}{a}\int_{S_{at_0}^{M_0}} r\psi \, d\Omega\right]_{r=at_0} . \end{aligned} \tag{5-2.4}$$

Hence, if the index 0 is dropped from $M_0 t_0$, we obtain the Poisson formula

$$u(M, t) = \frac{1}{4\pi a}\left[\frac{\partial}{\partial t}\int_{S_{at}^{M}}\frac{\varphi}{r} dS + \int_{S_{at}^{M}}\frac{\psi}{r} dS\right] .$$

As a second example, we consider the solution of the inhomogeneous wave equation with homogeneous initial conditions. We again set $S = S_{at_0}^{M_0}$ and note that the surface integral in formula (5-2.3) is equal to zero. Consequently, we obtain

$$u(M_0, t_0) = \frac{1}{4\pi}\int\int_{T_{at_0}^{M_0}}\frac{[f]}{r} d\tau_M , \tag{5-2.5}$$

where $T_{at_0}^{M_0}$ is a sphere with radius at_0 about M_0. We shall investigate in detail the case in which the inhomogeneous part of the wave equation is a periodic function of time:

$$f(M, t) = f_0(M)e^{i\omega t} .$$

Therefore let ω denote a given frequency. From Eq. (5-2.5) we obtain

$$u(M_0, t_0) = e^{i\omega t_0}\frac{1}{4\pi}\iint_{T_{at_0}^{M_0}}\int f_0(M)\frac{e^{-ikr}}{r}\,d\tau_M , \qquad \left(r = r_{MM_0},\ k = \frac{\omega}{a}\right). \tag{5-2.6}$$

Now let us consider the function $f_0(M)$, which is different from 0 only in the interior of a certain spatial region T. If M_0 lies in the exterior of T and if the distance between M_0 and the nearest point of region T is equal to d, then the integral over T_{at_0} for $t_0 < d/a$ is equal to 0. For such values the excitation has still not reached the point M_0. If the distance between M_0 and the furthest point of T is equal to D, then for $t_0 > D/a$, the integral on the right side is constant and reduces to an integral which extends over the entire region T. Consequently, from the time $t_0 = D/a$ on, at every point M_0, a periodic vibration occurs with amplitude given by

$$v(M_0) = \frac{1}{4\pi}\iint_T\int f_0(M)\frac{e^{ikr}}{r}\,d\tau_M , \qquad (r = r_{MM_0}) , \tag{5-2.7}$$

so that

$$u(M_0, t_0) = v(M_0)e^{i\omega t_0} .$$

By insertion of expression (5-2.6) for u (for $t_0 > D/a$) in the vibration equation, it follows that the function $v(M)$ must satisfy the differential equation

$$\Delta v + k^2 v = -f_0(M) , \qquad \left(k > \frac{\omega}{a}\right), \tag{5-2.8}$$

which we call the wave equation in the following discussion and which we shall investigate further in the Appendix.

We consider now formula (5-2.3) for the case of forced vibrations where

$$u(M, t) = v(M)e^{i\omega t} .$$

Let $v(M)$ be the amplitude of vibration, and let it satisfy Eq. (5-2.8).

Here, we have

$$[u] = u\left(M, t - \frac{r_{MM_0}}{a}\right) = v(M)e^{i(\omega t - kr)} , \qquad \left[\frac{\partial u}{\partial n}\right] = \frac{\partial v}{\partial n}e^{i(\omega t - kr)} ,$$

$$\frac{1}{a}\left[\frac{\partial u}{\partial t}\right] = ikv(M)e^{i(\omega t - kr)} , \qquad [f] = f_0(M)e^{i(\omega t - kr)} .$$

If we substitute these expressions into formula (5-2.3), then we arrive at the integral formula for the wave equation,

$$v(M_0) = \frac{1}{4\pi}\int_S\int\left[\frac{e^{-ikr}}{r}\frac{\partial v}{\partial n} - v\frac{\partial}{\partial n}\left(\frac{e^{-ikr}}{r}\right)\right]dS_M$$
$$+ \frac{1}{4\pi}\iint_T\int f_0(M)\frac{e^{-ikr}}{r}\,d\tau_M , \qquad (r = r_{MM_0}) , \tag{5-2.9}$$

which is usually called the Kirchhoff formula.

For $k = 0$, formula (5-2.9) is transformed into Green's formula (Section 4-2) for the inhomogeneous Laplace differential equation.

5-3. VIBRATIONS OF A BOUNDED SPATIAL REGION

1. A general scheme for the method of separation of variables. Standing waves

The problem of the vibrations of bounded spatial regions can be formulated in the following manner:

Determine the solution of the equation

$$\operatorname{div}(k \operatorname{grad} u) - q(M)u = \rho(M)u_{tt} \tag{5-3.1}$$

[the point $M = M(x, y, z)$ traverses the interior of a fixed spatial region T which is bounded by a closed surface S] which satisfies the following auxiliary conditions:

$$\left.\begin{aligned} u(M, 0) &= \varphi(M) \\ u_t(M, 0) &= \psi(M) \end{aligned}\right\} \quad \text{in the interior of } T \tag{5-3.2}$$

$$u|_S = 0 \quad \text{for } t > 0\,. \tag{5-3.3}$$

In the case of a homogeneous medium (k = const, ρ = const), Eq. (5-3.1), for $q = 0$, takes the form

$$Lu = \frac{1}{a^2}u_{tt} - f\,, \qquad \left(a^2 = \frac{k}{\rho}\,,\ f = \frac{F}{k}\right).$$

We are led to this and similar problems when studying, for example, vibrations of a membrane (two independent position variables), acoustic vibrations of a gas, and electromagnetic processes in nonconductors. Of special significance are those problems which are related to forced electromagnetic vibrations in cavity resonators (endovibrators, klystrons, magnetrons, and so forth).

Let us note that the homogenity of the boundary condition (5-3.3) signifies no loss of generality.

The case

$$u|_S = \mu\,, \tag{5-3.3'}$$

where μ is an arbitrary function of the point P on the surface S and the time t, can very easily be incorporated in the case for the homogeneous boundary condition and, indeed, has already been explained for one variable in Section 2-3. There we considered the difference of the prescribed function; correspondingly, this is true of the second and third boundary-value problems.

We shall now determine the solution $u(M, t)$ of the homogeneous equation (5-3.1) under the conditions (5-3.2) and (5-3.3) by separation of variables. We shall limit ourselves, therefore, to the investigation of some formal solution schemes. For this purpose, we first consider the following auxiliary problem (see Section 2-3):

Determine the nontrivial solution of the homogeneous differential equation

$$\operatorname{div}(k \operatorname{grad} u) - qu = \rho u_{tt} , \tag{5-3.1'}$$

with the boundary conditions

$$u|_S = 0 , \tag{5-3.3}$$

which are representable as a product, that is, in the form

$$u(M, t) = v(M)T(t) . \tag{5-3.4}$$

If we substitute Eq. (5-3.4) into Eq. (5-3.1) and separate variables in the usual way, we obtain the following equations for the functions $v(M)$ and $T(t)$:

$$\operatorname{div}(k \operatorname{grad} v) - qv + \lambda\rho v = 0 , \qquad v|_S = 0 , \tag{5-3.5}$$

$$T'' + \lambda T = 0 . \tag{5-3.6}$$

The differential equation (5-3.5) represents an eigenvalue problem; that is, we must determine the eigenvalue λ, which corresponds to the nontrivial solutions (the eigenfunctions) of the homogeneous equation with homogeneous boundary conditions.

We shall now go into further details of this problem which is analogous to that in Section 2-3. In our case, the equation for the eigenfunctions is a partial differential equation. As a result, the derivation of an explicit representation of the eigenfunctions for an arbitrary region T presents difficulties. Therefore, we consider in Sections 5-3.2 and 5-3.3 a region T for which an explicit representation is possible, even if it becomes necessary to introduce a new class of special functions. At this point we shall specify only general properties of eigenfunctions and eigenvalues and use the formal scheme of separation of variables. We begin by enumerating these properties.

1. There exist countably many eigenvalues $\lambda_1 < \lambda_2 < \cdots < \lambda_n < \cdots$, which correspond to the eigenfunctions

$$v_1(x, y, z), v_2(x, y, z), \ldots, v_n(x, y, z), \ldots$$

The eigenvalues increase unboundedly with increasing index n: $\lambda_n \to \infty$ as $n \to \infty$.

2. For $q \geqq 0$ all eigenvalues λ_n are positive: $\lambda_n > 0$.

3. The eigenfunctions v_n are orthogonal to each other with respect to the weight function in the region T; $\rho(x, t, z)$:

$$\int_T v_m(M)v_n(M)\rho(M)\, d\tau_M = 0 , \qquad (m \neq n) ,$$
$$M = M(x, y, z) , \qquad d\tau_M = dx\, dy\, dz . \tag{5-3.7}$$

4. Expansion theorem: Every twice continuously differentiable function $F(M)$ which satisfies the boundary condition $F = 0$ on S can be expanded in a uniformly and absolutely convergent series of eigenfunctions $v_n(M)$:

$$F(M) = \sum_{n=1}^{\infty} F_n v_n(M) ,$$

where F_n are the coefficients of expansion.

The proofs of properties 1 and 4 are usually carried out by using the theory of integral equations. We shall be content with the proofs of properties 2 and 3 which require no special apparatus. First we show the orthogonality of the eigenfunctions v_n (property 3). Let $y_n(M)$ and $v_m(M)$ be two eigenfunctions which correspond to different eigenvalues λ_n and λ_m. Then

$$\frac{\partial}{\partial x}\left(k\frac{\partial v_m}{\partial x}\right) + \frac{\partial}{\partial y}\left(k\frac{\partial v_m}{\partial y}\right) + \frac{\partial}{\partial z}\left(k\frac{\partial v_m}{\partial z}\right) - qv_m + \lambda_m \rho v_m = 0 ,$$

$$\frac{\partial}{\partial x}\left(k\frac{\partial v_n}{\partial x}\right) + \frac{\partial}{\partial y}\left(k\frac{\partial v_n}{\partial y}\right) + \frac{\partial}{\partial z}\left(k\frac{\partial v_n}{\partial z}\right) - qv_n + \lambda_n \rho v_n = 0 ,$$

where $v_m = 0$ and $v_n = 0$ on S. Multiplying the first equation by $v_n(M)$, the second equation by $v_m(M)$, and subtracting the second equation so obtained from the first, we have

$$\iiint_T \{v_n \operatorname{div}(k \operatorname{grad} v_m) - v_m \operatorname{div}(k \operatorname{grad} v_n)\}\, d\tau + (\lambda_m - \lambda_n)\iiint_T v_n v_m \rho\, d\tau = 0 .$$

Hence, after some transformations similar to those which were carried out for the derivation of the second Green's formula,[6] we obtain

$$\iint_S \left(v_n k\frac{\partial v_m}{\partial \nu} - v_m k\frac{\partial v_n}{\partial \nu}\right) d\sigma + (\lambda_m - \lambda_n)\iiint_T v_m v_n \rho\, d\tau = 0 ,$$

where ν denotes the direction of the exterior normal. From the boundary conditions, however, $v_m = 0$ and $v_n = 0$ on S, so that

$$(\lambda_m - \lambda_n)\iiint_T v_m v_n \rho\, d\tau = 0 .$$

Hence, for $\lambda_m \neq \lambda_n$ there results

$$\iiint_T v_m v_n \rho\, d\tau = 0 , \qquad (m \neq n) ;$$

that is, two eigenfunctions which correspond to different eigenvalues are orthogonal to each other with respect to the weight function $\rho(M)$.

In the investigation of the corresponding boundary-value problem for one independent variable

$$X'' + \lambda \rho X = 0 , \qquad X(0) = 0 , \qquad X(l) = 0 ,$$

we have shown that each eigenvalue corresponds exactly to one normalized eigenfunction. For two and three variables this is no longer the case. It will become clear from the examples to be given later that for eigenfunctions of a rectangle or a circle (see below) one and the same eigenvalue can correspond to several eigenfunctions. However, it is known from the theory of integral equations, that to every eigenvalue there belongs at most a finite number of linearly independent eigenfunctions.

[6] In this formula, the normal derivatives of the eigenfunctions are on the surface S. For the derivation of this formula for Liapunov surfaces, see V. I. Smirnov, *Textbook of Higher Mathematics*, Part IV, VEB Deutscher Verlag der Wissenschaften, Berlin, 1958.

Now let the eigenvalue λ_n correspond to the system of linearly independent eigenfunctions $v_n^{(1)}, v_n^{(2)}, \ldots, v_n^{(m)}$. Obviously, then, every linear combination of these functions

$$\bar{v}_n = \sum_{i=1}^{m} \alpha_i v_n^{(i)}$$

is also an eigenfunction with the same eigenvalue λ_n. With the help of the well-known Schmidt orthogonalization process,[7] we can construct an orthogonal set of functions $\bar{v}_n^{(1)}, \ldots, \bar{v}_n^{(m)}$ which are linear combinations of the original eigenfunctions. Therefore, the eigenfunctions belonging to the eigenvalue λ_n, which are not orthogonal to each other, can be transformed by orthogonalization to a system of eigenfunctions which are orthogonal to each other and correspond to λ_n.

The totality of this system of eigenfunctions for different λ_n forms an orthogonal system of eigenfunctions of the boundary-value problem under consideration

$$\operatorname{div}(k \operatorname{grad} v) - qv + \lambda \rho v = 0 , \qquad v = 0 \text{ on } S.$$

The expression

$$\sqrt{N_n} = \sqrt{\iiint_T v_n^2 \rho \, d\tau} > 0$$

is called the norm of the eigenfunction v_n. By multiplication of v_n with $1/\sqrt{N_n}$, we obtain a system of normalized eigenfunctions (that is, eigenfunctions whose norms are all equal to one).

For the proof of property 2, it is sufficient to apply Green's first formula

$$\begin{aligned} \iiint_T (k \operatorname{grad} v_n)^2 \, d\tau &= -\iiint_T v_n \operatorname{div}(k \operatorname{grad} v_n) \, d\tau + \iint_S v_n k \frac{\partial v_n}{\partial n} \, d\sigma \\ &= -\iiint_T q v_n^2 \, d\tau + \lambda_n \iiint_T v_n^2 \rho \, d\tau . \end{aligned}$$

From this formula it immediately follows that for $q \geqq 0$, the eigenvalues λ_n are positive.

We shall now make use of the expansion theorem (property 4) whose proof we assume is known from the theory of integral equations. Therefore let

$$F(M) = \sum_{n=1}^{\infty} F_n v_n(M) .$$

Then, in the usual manner, by using the orthogonality relation (5-3.7) for the expansion coefficients, we obtain

$$F_n = \frac{\iiint_T F(M) v_n(M) \rho \, d\tau}{\iiint_T v_n^2 \rho \, d\tau} . \tag{5-3.8}$$

We turn again to our partial differential equation. The solution of the differential equation

[7] See, for example, V. I. Smirnov, *Textbook of Higher Mathematics*, Part IV, VEB Deutscher Verlag der Wissenschaften, Berlin, 1958.

$$T_n'' + \lambda_n T_n = 0$$

has the form

$$T_n(t) = A_n \cos \sqrt{\lambda_n} t + B_n \sin \sqrt{\lambda_n} t ,$$

so that the solution of our auxiliary problem has the form

$$u_n(M, t) = T_n(t) v_n(M) = (A_n \cos \sqrt{\lambda_n} t + B_n \sin \sqrt{\lambda_n} t) v_n(M) .$$

The general solution of the original initial value problem is written in the form

$$u(M, t) = \sum_{n=1}^{\infty} u_n(M, t) = \sum_{n=1}^{\infty} (A_n \cos \sqrt{\lambda_n} t + B_n \sin \sqrt{\lambda_n} t) v_n(M) . \qquad (5\text{-}3.9)$$

If these are to satisfy the initial conditions (5-3.2).

$$u(M, 0) = \varphi(M) = \sum_{n=1}^{\infty} A_n v_n(M) ,$$

$$u_t(M, 0) = \psi(M) = \sum_{n=1}^{\infty} B_n \sqrt{\lambda_n} v_n(M) ,$$

then, as we know by using the expansion theorem, we must have

$$A_n = \varphi_n , \qquad B_n \sqrt{\lambda_n} = \psi_n ,$$

where φ_n and ψ_n are the Fourier coefficients of the functions $\varphi(M)$ or $\psi(M)$ in their expansion in terms of the orthogonal system of eigenfunctions $v_n(M)$ with weight $\rho(M)$. Thus the formal construction of the solution of the initial problem is completed.

The physical interpretation of the solution obtained is completely analogous to the case of one variable. The individual solutions

$$u_n(M, t) = (A_n \cos \sqrt{\lambda_n} t + B_n \sin \sqrt{\lambda_n} t) v_n(M)$$

represent standing waves which can occur in the interior of a bounded spatial region T.

The "profile" of the standing wave, defined by the function $v_n(M)$, can differ at different times only by a proportionality factor. Lines or surfaces (for two or three variables) on which $v_n(M) = 0$, are called nodal lines (surfaces) of the standing waves $v_n(M)$. The points at which $v_n(M)$ assumes a relative maximum or minimum are called maxima of the standing waves considered.[8]

The general solution can be represented as an infinite sum of such standing waves. The possibility of this representation means that every vibration can be obtained by superposition of the standing waves.[9]

Consequently, the problem of the vibrations of a membrane or a spatial region essentially leads to the determination of the corresponding eigenfunc-

[8] If a membrane is strewn with a fine powder and set vibrating, the powder collects in the nodal lines, whereby the so-called Chladni sound figures result. These reproduce the nodal lines of the eigenfunctions.

[9] The method of separation of variables in the case of several variables was established in the work of O. A. Ladyshenskaya, *Doklady*, 85, 3, 1952, p. 481.

tions. In the following two sections we shall investigate the vibrations of rectangular and circular membranes. Therefore, we shall direct our attention principally to the construction of the eigenfunctions. As already noted above, the determination of the eigenfunctions in an explicit analytic representation, when we are dealing with a complicated region, causes great difficulties. For an arbitrary region, approximation methods can be used for the construction of eigenfunctions; indeed, there exist different methods which depend on the use of integral equations, variational principles, or difference methods.[10]

2. Vibrations of a rectangular membrane

The vibrations of a plane homogeneous membrane, as represented in Section 2-2, is described by the wave equation

$$u_{tt} = a^2 \Delta u \,. \tag{5-3.10}$$

Let the rectangular membrane under consideration be in the x, y-plane, let it have sides of length b_1 and b_2, and let it be fixed along the boundary. Through an initial displacement and an initial velocity, the membrane is made to vibrate. For the determination of the function $u(x, y, t)$ which describes the displacement of the membrane from the equilibrium position, we must solve the wave equation

$$\frac{\partial^2 u}{\partial t^2} = a^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right), \tag{5-3.10'}$$

with the initial conditions

$$u(x, y, 0) = \varphi(x, y) \,, \qquad \frac{\partial u}{\partial t}(x, y, 0) = \psi(x, y) \,, \tag{5-3.11}$$

and the boundary conditions

$$u(0, y, t) = 0 \,, \qquad u(b_1, y, t) = 0 \,, \tag{5-3.12}$$

$$u(x, 0, t) = 0 \,, \qquad u(x, b_2, t) = 0 \,. \tag{5-3.13}$$

As usual, we try to determine the solution with the help of the method of separation of variables; we make the Ansatz

$$u(x, y, t) = v(x, y)T(t) \,. \tag{5-3.14}$$

By putting Eq. (5-3.14) into Eq. (5-3.10) and separating the variables, we obtain for the function $T(t)$ the differential equation

$$T'' + a^2 \lambda T = 0 \,, \tag{5-3.15}$$

and for the function $v(x, y)$ the boundary-value problem

$$\begin{aligned} v_{xx} + v_{yy} + \lambda v &= 0 \,, \\ v(0, y) = 0 \,, \qquad v(b_1, y) &= 0 \,, \\ v(x, 0) = 0 \,, \qquad v(x, b_2) &= 0 \,. \end{aligned} \tag{5-3.16}$$

[10] See, for example, Kantorovich-Krylov, *Approximate Methods of Higher Analysis*, Berlin, 1956. English translation: Interscience Publ., Inc., New York; P. Noordhoff Ltd., Groningen, The Netherlands, 1956.

The eigenvalue problem, therefore, consists of the solution of a homogeneous partial differential equation with homogeneous boundary conditions. This problem can also be solved by separation of variables. Therefore, we set $v(x, y) = X(x)Y(y)$.

For the functions $X(x)$ and $Y(y)$, we have the eigenvalue problems (in one independent variable)

$$X'' + \nu X = 0\,, \qquad X(0) = 0\,, \qquad X(b_1) = 0\,; \tag{5-3.17}$$

$$Y'' + \mu Y = 0\,, \qquad Y(0) = 0\,, \qquad Y(b_2) = 0\,. \tag{5-3.18}$$

Here ν and μ are constants which arise from separation of variables and are connected with each other by the relation $\mu + \nu = \lambda$. The boundary conditions for the functions $X(x)$ and $Y(y)$ result from the corresponding conditions for the function v. For example, the condition $X(0) = 0$ follows from

$$v(0, y) = X(0)Y(y) = 0\,,$$

since $Y(y) \neq 0$. (We naturally consider nontrivial solutions.)

We have already discussed the solution of problems similar in nature to problems (5-3.17) and (5-3.18), when we investigated the vibrating string. The solutions of problems (5-3.17) and (5-3.18) have the form

$$X_n(x) = \sin\frac{n\pi}{b_1}x\,, \qquad Y_m(y) = \sin\frac{m\pi}{b_2}y\,,$$

$$\nu_n = \left(\frac{n\pi}{b_1}\right)^2, \qquad \mu_m = \left(\frac{m\pi}{b_2}\right)^2.$$

The eigenvalues

$$\lambda_{n,m} = \left(\frac{n\pi}{b_1}\right)^2 + \left(\frac{m\pi}{b_2}\right)^2$$

correspond, therefore, to the eigenfunctions

$$v_{n,m} = A_{n,m}\sin\frac{n\pi}{b_1}x\sin\frac{m\pi}{b_2}y\,,$$

where $A_{n,m}$ are arbitrary constant factors. We choose these factors so that the norm of the functions $v_{n,m}$ with weight 1 are equal to 1:

$$\int_0^{b_1}\int_0^{b_2} v_{n,m}^2\,dx\,dy = A_{n,m}^2\int_0^{b_1}\sin^2\frac{n\pi}{b_1}x\,dx\int_0^{b_2}\sin^2\frac{m\pi}{b_2}y\,dy = 1\,.$$

From this follows

$$A_{n,m} = \sqrt{\frac{4}{b_1 b_2}}\,.$$

The orthogonality of the functions $v_{n,m}$ obviously require no proof. Therefore, the functions

$$v_{n,m}(x, y) = \sqrt{\frac{4}{b_1 b_2}}\sin\frac{n\pi}{b_1}x\sin\frac{m\pi}{b_2}y \tag{5-3.19}$$

form an orthonormal system of eigenfunctions for the rectangular membrane.

The number of eigenfunctions belonging to $\lambda_{n,m}$ (the multiplicity of $\lambda_{n,m}$) depends on n and m, the number of integer solutions of the equation

$$\left(\frac{n\pi}{b_1}\right)^2 + \left(\frac{m\pi}{b_2}\right)^2 \quad \lambda_{n,m} .$$

The system of eigenfunctions $v_{n,m}$ thus determined has the property that every twice continuously differentiable function which satisfies the corresponding boundary conditions can be expanded in an absolutely and uniformly convergent series with respect to the functions $v_{n,m}$. This assertion is proved in the theory of multiple Fourier series.

Now we shall show that the system (5-3.19) contains all eigenfunctions of our eigenvalue problems. Let us assume that there exists still another eigenfunction u_0. Let this belong to the eigenvalue λ_0. Since u_0 is orthogonal to all eigenfunctions which correspond to other eigenvalues λ, there remains in the expansion of u_0 with respect to the system (5-3.19) only a finite number of members. These terms correspond to the eigenfunctions of (5-3.19) which belong to the eigenvalue $\lambda_{n,m} = \lambda_0$. Therefore, u_0 is a linear combination of functions which belong entirely to the system (5-3.19) and correspond to the eigenvalue $\lambda_{n,m} = \lambda_0$. Thus all eigenfunctions of the rectangular membrane are contained in (5-3.19).

If we now return to the original problem for Eq. (5-3.10), we see that the individual solutions

$$u_{n,m} = v_{n,m}(\bar{B}_{n,m} \cos \sqrt{\lambda_{n,m}}\, at + B_{n,m} \sin \sqrt{\lambda_{n,m}}\, at)$$

represent standing waves whose profile is determined by the eigenfunctions $v_{n,m}$. The geometric locus of the points within the rectangle at which the eigenfunctions vanish are the nodal lines. For simplicity, we consider a square with sides of length $b(b_1 \quad b_2)$. The nodal lines of the functions

$$v_{n,m} = \frac{2}{b} \sin \frac{n\pi}{b} x \sin \frac{m\pi}{b} y$$

are then straight lines which are parallel to the coordinate axes (Figure 81 a).

In the case of multiple eigenvalues, the linear combinations of the cor-

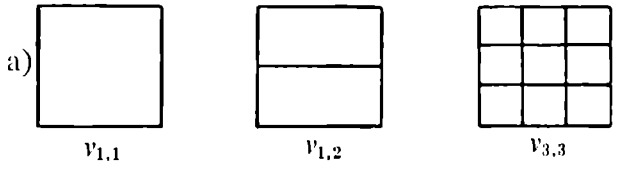


FIG. 81 a

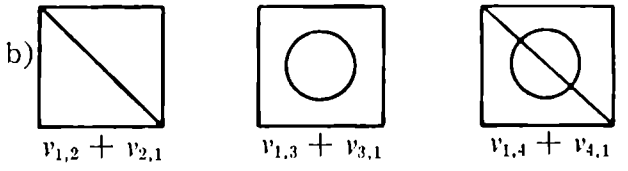


FIG. 81 b

responding eigenfunctions are likewise eigenfunctions whose nodal lines can be of a very complicated nature (Figure 81 b).

The desired solution of Eq. (5-3.10) with auxiliary conditions (5-3.11) to (5-3.13) has the form

$$u(x, y, t) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} (\bar{B}_{n,m} \cos \sqrt{\lambda_{n,m}}\, at + B_{n,m} \sin \sqrt{\lambda_{n,m}}\, at) v_{n,m}(x, y) .$$

Hence the $v_{n,m}$ are determined by the formula (5-3.19), while the coefficients $\bar{B}_{n,m}$ and $\bar{\bar{B}}_{n,m}$ are calculated in the following manner:

$$\bar{B}_{n,m} = \int_0^{b_1}\int_0^{b_2} \varphi(x, y) v_{n,m}(x, y)\, dx\, dy = \sqrt{\frac{4}{b_1 b_2}} \int_0^{b_1}\int_0^{b_2} \varphi(x, y) \sin \frac{n\pi}{b_1} x \sin \frac{m\pi}{b_2} y\, dx\, dy ,$$

$$B_{n,m} = \frac{1}{\sqrt{a^2 \lambda_{n,m}}} \sqrt{\frac{4}{b_1 b_2}} \int_0^{b_1}\int_0^{b_2} \psi(x, y) \sin \frac{n\pi}{b_1} x \sin \frac{m\pi}{b_2} y\, dx\, dy .$$

3. Vibrations of a circular membrane

To investigate the vibrations of a circular membrane we use polar coordinates. In these coordinates the wave equation has the form

$$\frac{1}{r} \frac{\partial}{\partial r}\left(r \frac{\partial u}{\partial r}\right) + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} = \frac{1}{a^2} \frac{\partial^2 u}{\partial t^2} . \tag{5-3.20}$$

We shall seek a solution of this equation which satisfies the initial conditions

$$u(r, \theta, 0) = f_1(r, \theta) , \qquad u_t(r, \theta, 0) = f_2(r, \theta) \tag{5-3.21}$$

and the boundary condition

$$u(r_0, \theta, t) = 0 ; \tag{5-3.22}$$

that is, we shall consider a circular membrane of radius r_0 fixed along the length of the boundary. As in the case of a rectangular membrane, we shall also determine the solution by separation of variables. By the Ansatz

$$u(r, \theta, t) = v(r, \theta) T(t) ,$$

there results for $T(t)$ the differential equation

$$T'' + a^2 \lambda T = 0 ,$$

$$T = C_1 \cos \sqrt{a^2 \lambda}\, t + C_2 \sin \sqrt{a^2 \lambda}\, t ,$$

while for $v(r, \theta)$ the following eigenvalue problem arises:

$$\frac{1}{r} \frac{\partial}{\partial r}\left(r \frac{\partial v}{\partial r}\right) + \frac{1}{r^2} \frac{\partial^2 v}{\partial \theta^2} + \lambda v = 0 , \qquad (0 < r < r_0) ,$$

$|v(0, \theta)| < \infty$ Boundedness condition,

$v(r_0, \theta) = 0$ Boundary condition,

$v(r, \theta) = v(r, \theta + 2\pi)$ Periodicity condition.

The function $v(r, \theta)$ must be a single valued and differentiable function of position; however, since θ is a cyclic coordinate, we must require the perio-

dicity of v with period 2π, i. e., $v(r, \theta + 2\pi) = v(r, \theta)$ in order to guarantee the single valuedness of $v(r, \theta)$.

Now again we set $v(r, \theta) = R(r)\Theta(\theta)$. If, in the above equation, we use the Ansatz and divide by $R\Theta$, we obtain

$$\frac{r\frac{d}{dr}\left(r\frac{dR}{dr}\right)}{R} + \frac{\Theta''}{\Theta} + \lambda r^2 = 0 .$$

From this we obtain the equations

$$\Theta'' + \mu^2\Theta = 0 ,$$
$$\Theta(\theta) = \Theta(\theta + 2\pi) , \qquad \Theta'(\theta) = \Theta'(\theta + 2\pi) ,$$
$$\frac{1}{r}\frac{d}{dr}\left(r\frac{dR}{dr}\right) + \left(\lambda - \frac{\mu^2}{r^2}\right)R = 0 ,$$
$$R(r_0) = 0 , \qquad |R(0)| < \infty .$$

The nontrivial periodic solutions $\Theta(\theta)$ exist only for $\mu^2 = n^2$ (n is an integer); it has the form

$$\Theta_n(\theta) = D_{1n}\cos n\theta + D_{2n}\sin n\theta .$$

We note that two linearly independent eigenfunctions belong to the eigenvalue n^2, namely, $\cos n\theta$ and $\sin n\theta$.

For the determination of the function $R(r)$, we have to solve the equation

$$\frac{d^2R}{dr^2} + \frac{1}{r}\frac{dR}{dr} + \left(\lambda - \frac{n^2}{r^2}\right)R = 0 \tag{5-3.23}$$

with the homogeneous boundary conditions

$$R(r_0) = 0 , \qquad |R(0)| < \infty . \tag{5-3.24}$$

The determination of $R(r)$, therefore, amounts to solving these eigenvalue problems.

The second requirement placed on $R(r)$, namely, that for $r = 0$ it must be bounded, is related to the fact that $r = 0$ is a singular point of the equation. For singular points it is sufficient to require boundedness as a boundary condition (see Section A-2.3).

We now introduce the new variable

$$x = \sqrt{\lambda}\, r$$

and write

$$R(r) = R\left(\frac{x}{\sqrt{\lambda}}\right) = y(x) .$$

For the determination of $y(x)$ we then obtain the equation for the cylindrical functions of the nth order (see Appendix).

$$\frac{d^2y}{dx^2} + \frac{1}{x}\frac{dy}{dx} + \left(1 - \frac{n^2}{x^2}\right)y = 0 \tag{5-3.25}$$

with the boundary conditions

$$y(x_0) = 0 , \qquad (x_0 = \sqrt{\lambda}\, r_0) , \qquad |y(0)| < \infty . \tag{5-3.26}$$

The general solution of the equation for cylindrical functions has the form

$$y(x) = d_1 J_n(x) + d_2 N_n(x) , \tag{5-3.27}$$

where $J_n(x)$ is the Bessel function of the nth order and N_n is the Neumann function of the nth order (see Appendix). Then $d_2 = 0$ follows from the second condition. The first condition yields

$$J_n(\sqrt{\lambda}\, r_0) = 0 \qquad \text{or} \qquad J_n(\mu) = 0 \qquad (\mu = \sqrt{\lambda}\, r_0) . \tag{5-3.28}$$

If $\mu_m^{(n)}$ is the mth root of the equation $J_n(\mu) = 0$, then

$$\lambda_{n,m} = \left(\frac{\mu_m^{(n)}}{r_0}\right)^2 . \tag{5-3.29}$$

To this eigenvalue belongs the eigenfunction

$$R_{n,m} = y(\sqrt{\lambda}\, r) = J_n\left(\frac{\mu_m^{(n)}}{r_0} r\right) . \tag{5-3.30}$$

The eigenfunction (5-3.30) posseses the following properties:[11]

1. The eigenfunctions $R(r)$ which belong to different eigenvalues λ, are orthogonal to each other with weight r;

$$\int_0^{r_0} r R_{n,m_1}(r) R_{n,m_2}(r)\, dr = 0 , \qquad (m_1 \neq m_2) ,$$

or

$$\int_0^{r_0} r J_n\left(\frac{\mu_{m_1}^{(n)}}{r_0} r\right) J_n\left(\frac{\mu_{m_2}^{(n)}}{r_0} r\right) dr = 0 . \tag{5-3.31}$$

2. The norm of these functions is defined by

$$\int_0^{r_0} r J_n^2\left(\frac{\mu_m^{(n)}}{r_0} r\right) dr = \frac{r_0^2}{2} [J_n'(\mu_m^{(n)})]^2 . \tag{5-3.31'}$$

In particular, for the norm of the function $J_0((\mu_m^{(0)}/r_0)r)$ we have

$$\int_0^{r_0} r \left[J_0\left(\frac{\mu_m^{(0)}}{r_0} r\right)\right]^2 dr = \frac{r_0^2}{2} [J_1(\mu_m^{(0)})]^2 . \tag{5-3.32}$$

3. Every function $f(r)$ which is continuous in the interval $(0, r_0)$ and which possesses there piecewise continuous first and second derivatives and satisfies the boundary conditions of our problem, can be expanded in an absolutely and uniformly convergent series

$$f(r) = \sum_{m=1}^{\infty} f_m J_n\left(\frac{\mu_m^{(n)}}{r_0} r\right) , \tag{5-3.33}$$

where the expansion coefficients are given by

$$f_m = \frac{\displaystyle\int_0^{r_0} r f(r) J_n\left(\frac{\mu_m^{(n)}}{r_0} r\right) dr}{\dfrac{r_0^2}{2} [J_n'(\mu_m^{(n)})]^2} . \tag{5-3.34}$$

[11] See Section A-2.1.

We now turn again to the eigenvalue problem for the circular membrane. Its eigenvalues are

$$\lambda_{n,m} = \left(\frac{\mu_m^{(n)}}{r_0}\right)^2 ,$$

and each of these eigenvalues corresponds to the two eigenfunctions

$$\bar{v}_{n,m} = J_n\left(\frac{\mu_m^{(n)}}{r_0} r\right)\cos n\theta , \qquad \bar{\bar{v}}_{n,m} = J_n\left(\frac{\mu_m^{(n)}}{r_0} r\right)\sin n\theta . \tag{5-3.35}$$

By a linear combination of such eigenfunctions, there results

$$v_{n,m}(r, \theta) = J_n\left(\frac{\mu_m^{(n)}}{r_0} r\right)(A_{n,m}\cos n\theta + B_{n,m}\sin n\theta) . \tag{5-3.36}$$

We shall now calculate the norm of the eigenfunction $v_{n,m}$; the orthogonality of the eigenfunctions follows from the general theory. For simplicity, we shall limit ourselves to the calculation of the eigenfunctions $\bar{v}_{n,m}$.

$$\begin{aligned}
&\int_0^{2\pi}\int_0^{r_0} \bar{v}_{n_1,m_1}\bar{v}_{n_2,m_2} r\, dr\, d\theta \\
&= \int_0^{r_0} J_{n_1}\left(\frac{\mu_{m_1}^{(n_1)}}{r_0} r\right) J_{n_2}\left(\frac{\mu_{m_2}^{(n_2)}}{r_0} r\right) r\, dr \int_0^{2\pi} \cos n_1\theta \cos n_2\theta\, d\theta \\
&\quad \begin{cases} 0 & \text{for } n_1 \neq n_2 , \\ 0 & \text{for } n_1 = n_2 , \quad m_1 \neq m_2 , \end{cases} \\
&= \begin{cases} \dfrac{r_0^2}{2}[J_n'(\mu_m^{(n)})]^2\pi & \text{for } n_1 = n_2 = n \neq 0 \text{ and } m_1 = m_2 = m , \\ \dfrac{r_0^2}{2}[J_0'(\mu_m^{(n)})]^2 2\pi & \text{for } n_1 = n_2 = 0 \quad \text{and } m_1 = m_2 = m . \end{cases}
\end{aligned} \tag{5-3.37}$$

The corresponding relations are obtained for the functions

$$v_{n,m} = J_n\left(\frac{\mu_m^{(n)}}{r_0} r\right)\sin n\theta .$$

The expression for the norm of $v_{n,m}$ can be written in the form

$$\int_\Sigma\int v_{n,m}^2\, d\sigma = \frac{r_0^2}{2}\pi\varepsilon_n[J_n'(\mu_m^{(n)})]^2 , \qquad (d\sigma = r\, dr\, d\theta) , \tag{5-3.38}$$

where

$$\varepsilon_n = \begin{cases} 2 & \text{for } n = 0 , \\ 1 & \text{for } n \neq 0 . \end{cases}$$

Furthermore, from the general theory we conclude:

Every continuous function $F(r, \theta)$ with continuous first and second derivatives satisfying the boundary conditions of our problem can be expanded in an absolutely and uniformly convergent sereies

$$F(r, \theta) = \sum_{n,m} (A_{n,m}\bar{v}_{n,m}(r, \theta) + B_{n,m}v_{n,m}(r, \theta)) \tag{5-3.39}$$

in terms of the eigenfunctions of the eigenvalue problems corresponding to the circle.

The expansion coefficients can be calculated from the formulas

$$A_{n,m} = \frac{\int_0^{2\pi}\int_0^{r_0} F(r,\theta) J_n\left(\frac{\mu_m^{(n)}}{r_0} r\right) \cos n\theta \, r \, dr \, d\theta}{\frac{\pi r_0^2}{2} \varepsilon_n [J_n'(\mu_m^{(n)})]^2},$$

$$B_{n,m} = \frac{\int_0^{2\pi}\int_0^{r_0} F(r,\theta) J_n\left(\frac{\mu_m^{(n)}}{r_0} r\right) \sin n\theta \, r \, dr \, d\theta}{\frac{\pi r_0^2}{2} [J_n'(\mu_m^{(n)})]^2}. \tag{5-3.40}$$

We still note that the function

$$v_{0,m}(r,\theta) = J_0\left(\frac{\mu_m^{(0)}}{r_0} r\right)$$

does not depend on θ.

If the prescribed function F depends only on r, say $F = F(r)$, then the expansion of F with respect to the eigenfunctions includes only the function $v_{0,m}$:

$$F(r) = \sum_{m=1}^{\infty} A_{0,m} J_0\left(\frac{\mu_m^{(0)}}{r_0} r\right), \tag{5-3.41}$$

where

$$A_{0,m} = \frac{\int_0^{r_0} F(r) J_0\left(\frac{\mu_m^{(0)}}{r_0} r\right) r \, dr}{\frac{r_0^2}{2} J_1^2(\mu_m^{(0)})}. \tag{5-3.42}$$

All of the remaining coefficients $A_{n,m}$ and $B_{n,m}$ are equal to zero.

The solution of our original problem, which was to determine the vibrations of a circular membrane with given initial displacement and initial velocity, can be written in the form

$$u(r,\theta,t) = \sum_{n,m=0}^{\infty} \bar{v}_{n,m}(r,\theta)\left(A_{n,m} \cos \frac{a\mu_m^{(n)}}{r_0} t + B_{n,m} \sin \frac{a\mu_m^{(n)}}{r_0} t\right)$$
$$+ \sum_{n,m=0}^{\infty} \bar{\bar{v}}_{n,m}(r,\theta)\left(C_{n,m} \cos \frac{a\mu_m^{(n)}}{r_0} t + D_{n,m} \sin \frac{a\mu_m^{(n)}}{r_0} t\right).$$

The coefficients $A_{n,m}$, $B_{n,m}$, $C_{n,m}$ and $D_{n,m}$ can be calculated from the initial conditions

$$u(r,\theta,0) = \sum_{n,m=0}^{\infty} (A_{n,m}\bar{v}_{n,m} + C_{n,m}\bar{\bar{v}}_{n,m}) = f_1(r,\theta),$$

$$u(r,\theta,0) = \sum_{n,m=0}^{\infty} (B_{n,m}\bar{v}_{n,m} + D_{n,m}\bar{\bar{v}}_{n,m}) \frac{a\mu_m^{(n)}}{r_0} = f_2(r,\theta)$$

according to the formulas

$$A_{n,m} = \frac{\int_0^{2\pi}\int_0^{r_0} \varphi(r,\theta) J_n(\sqrt{\lambda_{n,m}}r) \cos n\theta \, r \, dr \, d\theta}{\frac{\pi r_0^2}{2}\varepsilon_n |J_n'(\sqrt{\lambda_{n,m}}r)|^2},$$

$$C_{n,m} = \frac{\int_0^{2\pi}\int_0^{r_0} \varphi(r,\theta) J_n(\sqrt{\lambda_{n,m}}r) \sin n\theta \, r \, dr \, d\theta}{\frac{\pi r_0^2}{2}|J_n'(\sqrt{\lambda_{n,m}}r)|^2}.$$

Analogous formulas for $(a\sqrt{\lambda_{n,m}})B_{n,m}$ and, correspondingly, for $(a\sqrt{\lambda_{n,m}})D_{n,m}$ are valid.

Problems

1. Solve the initial value problem for the three-dimensional wave equation $u_{tt} = a^2 \Delta u$, assuming that the initial velocity is everywhere equal to 0, while the initial displacement $u|_{t=0} = \varphi$ has the following form:

(a) $\varphi = \begin{cases} 1 \text{ in the interior of a unit sphere,} \\ 0 \text{ in the exterior of the unit sphere;} \end{cases}$

(b) $\varphi = \begin{cases} A\cos((\pi/2r_0)r) \text{ in the interior of the sphere of radius } r_0, \\ 0 \text{ in the exterior of the sphere of radius } r_0. \end{cases}$

2. Determine the vibrations of the half space $z > 0$ for a homogeneous boundary condition of the first or second kind if we prescribe (a) a local excitation, that is, the initial velocity and initial displacement are in a fixed region T_0; (b) a concentrated force which acts according to an arbitrary law.

3. Solve the same problem for the slit $l' \leqq z \leqq l$.

4. Solve the wave equation for a region which forms a wedge with the vertex angle $\pi/2$ or π/n (n is an integer), if homogeneous boundary conditions of the first or second kind as well as initial velocity and initial displacement are prescribed.

5. Derive an analog of the integral formula (5-3.3) for the equation

$$u_{tt} = a^2 \Delta u + cu \quad \text{with } c = \text{const}.$$

Consider the case $c < 0$ and solve the initial value problem for the inhomogeneous equation. Give a physical interpretation of the results.

6. Determine the function $u(\rho, \varphi, t)$ which describes the vibrations of a membrane under the influence of a concentrated impulse K; this acts

(a) at the center of a circular membrane,

(b) at an arbitrary point of a circular membrane,

(c) at an arbitrary point of a rectangular membrane.

7. Determine the eigenfrequencies and eigenfunctions of a membrane in the form

(a) of a semicircle,

(b) of a ring,

(c) of a circular sector,

(d) of a ring sector.

Consider the first and second boundary-value problems.

8. Determine the forced vibrations of a circular membrane (e.g., membrane of a microphone) under the influence of a periodic force which is distributed on the membrane with constant density $f = A \sin \omega t$. Solve this problem for the case $f = A(1 - r^2 c^2) \sin \omega t$, where c denotes the radius of the membrane.

9. Derive the equation of sound propagation in a medium which moves with constant velocity. Convert the equation obtained by transforming to a coordinate system which moves with the medium.

5-4. APPLICATIONS OF CHAPTER 5

1. Reductions of the equations of the theory of elasticity to wave equations

The theory of elasticity is concerned with the problem of determining the deformation and motion occurring in elastic bodies with the help of mathematical methods. The deformations and motions resulting from the influence of an external force can be described by the displacement vector $\boldsymbol{u}$. Its projections on the three coordinate axes x, y, z, will be denoted by $u(x, y, z, t)$, $v(x, y, z, t)$ and $w(x, y, z, t)$, respectively. Displacements in an elastic body occur under the action of internal forces (stresses). These form a symmetric stress tensor denoted by

$$\begin{pmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_y & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_z \end{pmatrix}.$$

Here $\sigma_x, \tau_{xy}, \tau_{xz}$ are the components of the force per unit area (stresses) acting on the surface element perpendicular to the x-axis. Correspondingly, $\tau_{yx}, \sigma_y, \tau_{yz}$ and $\tau_{zx}, \tau_{zy}, \sigma_z$ are the components which act on the surface element perpendicular to the y- and z-axes, respectively. The components $\sigma_x, \sigma_y, \sigma_z$ are called normal stresses and the quantities $\tau_{xy}, \tau_{xz}, \ldots$ tangential or shear stresses. The equations of motion for an element of volume read

$$\begin{aligned} \rho \frac{\partial^2 u}{\partial t^2} &= \frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z} + X, \\ \rho \frac{\partial^2 v}{\partial t^2} &= \frac{\partial \tau_{yx}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{yz}}{\partial z} + Y, \\ \rho \frac{\partial^2 w}{\partial t^2} &= \frac{\partial \tau_{zx}}{\partial x} + \frac{\partial \tau_{zy}}{\partial y} + \frac{\partial \sigma_z}{\partial z} + Z, \end{aligned} \tag{5-4.1}$$

where ρ is the density at the point (x, y, z), while X, Y, Z denote the components of the external forces. The relation between the stresses and strains are described by Hooke's Law, which we can write as follows:

$$\begin{aligned} \sigma_x &= 2G\left\{\varepsilon_x + \frac{\theta}{m-2}\right\}, \qquad \tau_{xy} = G\gamma_{xy}, \\ \sigma_y &= 2G\left\{\varepsilon_y + \frac{\theta}{m-2}\right\}, \qquad \tau_{yz} = G\gamma_{yz}, \\ \sigma_z &= 2G\left\{\varepsilon_z + \frac{\theta}{m-2}\right\}, \qquad \tau_{zx} = G\gamma_{zx}, \end{aligned} \tag{5-4.2}$$

where the quantities

$$\varepsilon_x = \frac{\partial u}{\partial x}, \qquad \varepsilon_y = \frac{\partial v}{\partial y}, \qquad \varepsilon_z = \frac{\partial w}{\partial z},$$
$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}, \qquad \gamma_{yz} = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}, \qquad \gamma_{zx} = \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} \tag{5-4.3}$$

form a symmetric strain tensor denoted by

$$\begin{pmatrix} \varepsilon_x & \gamma_{xy} & \gamma_{xz} \\ \gamma_{yx} & \varepsilon_y & \gamma_{yz} \\ \gamma_{zx} & \gamma_{zy} & \varepsilon_z \end{pmatrix}.$$

Formula (5-4.2) has the following significance: G is the shear modulus and m the coefficient which characterizes the contraction of the body crosswise due to elongation of the body lengthwise. Further,

$$\theta = \operatorname{div} \boldsymbol{u} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}.$$

To Eqs. (5-4.1) and (5-4.2) we must still add the boundary conditions in question (on the boundary, we can prescribe, for example, the displacements u, v, w or surface forces), which we shall examine further here.

Equations (5-4.1) and (5-4.2) form a system of partial differential equations for the stresses and deformations. If we introduce the expression for the stresses from Eqs. (5-4.2) into Eqs. (5-4.1) and subtract Eqs. (5-4.3) from the result, we obtain the following system of equations for the displacements.

$$\begin{aligned} \rho \frac{\partial^2 u}{\partial t^2} &= G\left\{\Delta u + \frac{m}{m-2}\frac{\partial \theta}{\partial x}\right\} + X, \\ \rho \frac{\partial^2 v}{\partial t^2} &= G\left\{\Delta v + \frac{m}{m-2}\frac{\partial \theta}{\partial y}\right\} + Y, \\ \rho \frac{\partial^2 w}{\partial t^2} &= G\left\{\Delta w + \frac{m}{m-2}\frac{\partial \theta}{\partial z}\right\} + Z. \end{aligned} \tag{5-4.4}$$

Frequently, in place of the constants G and m, the so-called Lamé constants λ and μ are introduced. These are related to G and m through the relations

$$\mu = G, \qquad \lambda = \frac{2}{m-2}G.$$

With the help of the Lamé constants, the system of equations (5-4.4) can be written in the form of a single vector equation

$$\rho \frac{\partial \boldsymbol{u}}{\partial t^2} = (\lambda + 2\mu)\operatorname{grad} \operatorname{div} \boldsymbol{u} - \mu \operatorname{rot} \operatorname{rot} \boldsymbol{u} + \boldsymbol{F}. \tag{5-4.5}$$

Here $\boldsymbol{u}$ is the displacement vector with components u, v, w, and $\boldsymbol{F}$ is the vector of volume force with components X, Y, and Z.

Now we shall show that Eq. (5-4.5) can be reduced to two wave equations for suitably chosen functions.[12]

[12] P. Frank and R. V. Mises, *Differential and Integral Equations of Mechanics and Physics*, Braunschweig, 1927 1930, Chapter VII.

An arbitrary vector $\boldsymbol{F}$ can always be written in the form

$$\boldsymbol{F} = \operatorname{grad} U + \operatorname{rot} \boldsymbol{L} ,$$

where U is a scalar and $\boldsymbol{L}$ is a vector potential.

We now set

$$\boldsymbol{u} = \operatorname{grad} \Phi + \operatorname{rot} \boldsymbol{A} ,$$

where

$$\rho \frac{\partial^2 \Phi}{\partial t^2} = (\lambda + 2\mu) \Delta \Phi + U , \qquad \rho \frac{\partial^2 \boldsymbol{A}}{\partial t^2} = \mu \Delta \boldsymbol{A} + \boldsymbol{L}$$

holds.

By substitution, we can easily verify that the vector $\boldsymbol{u}$ so introduced actually satisfies the system of Eq. (5-4.4) of elasticity theory.

In the absence of volume forces, we then obtain for the potentials Φ and $\boldsymbol{A}$ the homogeneous wave equations

$$\rho \frac{\partial^2 \Phi}{\partial t^2} = (\lambda + 2\mu) \Delta \Phi , \qquad \rho \frac{\partial^2 \boldsymbol{A}}{\partial t^2} = \mu \Delta \boldsymbol{A} .$$

In certain cases, in a Cartesian coordinate system for example, the wave equation for the vector potential $\boldsymbol{A}$ separates into three scalar equations. However, the question of how to restore the equations of elasticity theory to single scalar wave equations without considering the boundary conditions is not completely answered, as the boundary conditions under certain conditions can couple the different components and make complete separation of the equations very difficult.

2. The equations of the electromagnetic field

1. *The equations of electromagnetic fields and boundary conditions.* An electromagnetic field is characterized by the electrical field vector $\boldsymbol{E}$ and the magnetic field vector $\boldsymbol{H}$ as well as by the vectors of electrical and magnetic induction $\boldsymbol{D}$ and $\boldsymbol{B}$. The complete system of Maxwell's equations relating these quantities reads

$$\operatorname{rot} \boldsymbol{H} = \frac{1}{c} \frac{\partial \boldsymbol{D}}{\partial t} + \frac{4\pi}{c} \boldsymbol{j} + \frac{4\pi}{c} \boldsymbol{j}^{(e)} , \tag{5-4.6}$$

$$\operatorname{rot} \boldsymbol{E} = -\frac{1}{c} \frac{\partial \boldsymbol{B}}{\partial t} , \tag{5-4.7}$$

$$\operatorname{div} \boldsymbol{B} = 0 , \tag{5-4.8}$$

$$\operatorname{div} \boldsymbol{D} = 4\pi\rho . \tag{5-4.9}$$

Here $\boldsymbol{j}$ is the current density of the convection current, $\boldsymbol{j}^{(e)}$ is the current density produced by the action of the given electromotive force, ρ is the charge density, and c is the velocity of light in a vacuum. In the following discussion, we shall assume $\boldsymbol{j}^{(e)} = 0$.

To these equations we must still add the so-called material or constitutive equations of the fields

$$\boldsymbol{D} = \varepsilon \boldsymbol{E} , \tag{5-4.10}$$

$$\boldsymbol{B} = \mu \boldsymbol{H} , \tag{5-4.11}$$

$$\boldsymbol{j} = \sigma \boldsymbol{E} . \tag{5-4.12}$$

Here ε denotes the dielectric constant, μ the magnetic permeability, and σ the conductivity of the medium. In the following discussion, we shall assume that the material under consideration is homogeneous and isotropic. Then, $\varepsilon = \text{const}$, $\mu = \text{const}$, $\sigma = \text{const}$. We often consider electromagnetic phenomena in a vacuum, where $\varepsilon = \mu = 1$, $\sigma = 0$, for no charges or currents present. In this case, Maxwell's equations simplify to

$$\text{rot } \boldsymbol{H} = \frac{1}{c}\frac{\partial \boldsymbol{E}}{\partial t} , \qquad \text{div } \boldsymbol{H} = 0 ,$$

$$\text{rot } \boldsymbol{E} = -\frac{1}{c}\frac{\partial \boldsymbol{H}}{\partial t} , \qquad \text{div } \boldsymbol{E} = 0 .$$

Equations (5-4.6) and (5-4.9) are consistent with each other since there exists between ρ and $\boldsymbol{j}$ the relation

$$\frac{\partial \rho}{\partial t} + \text{div } \boldsymbol{j} = 0 ,$$

which expresses the law of conservation of the charge.

The laws of electromagnetic fields, which are represented in differential form by Eqs. (5-4.6) to (5-4.9), can also be written in integral form

$$\oint_C H_s \, ds = \frac{4\pi}{c}\int_\Sigma \int i_n \, d\sigma , \tag{5-4.6'}$$

$$\oint_C E_s \, ds = -\frac{1}{c}\frac{d}{dt}\int_\Sigma \int B_n \, d\sigma = -\frac{1}{c}\frac{d\Phi}{dt} . \tag{5-4.7'}$$

Here

$$\boldsymbol{i} = \boldsymbol{j}_v + \boldsymbol{j} = \frac{1}{4\pi}\frac{\partial \boldsymbol{D}}{\partial t} + \boldsymbol{j} \tag{5-4.13}$$

is the total current, and $\boldsymbol{j}_v = (1/4\pi)(\partial \boldsymbol{D}/\partial t)$ is the displacement current. The integration is extended over a curve C or over a surface Σ which is bounded by this curve C; $\Phi = \int_\Sigma \int B_n \, d\sigma$ is the flux of induction passing through C. If T denotes any closed spatial region and Σ the bounding surface of T, then the equations

$$\int_\Sigma \int B_n \, d\sigma = 0 , \tag{5-4.8'}$$

$$\int_\Sigma \int D_n \, d\sigma = 4\pi \int\int_T \int \rho \, d\tau = 4\pi e \tag{5-4.9'}$$

replace Eqs. (5-4.8) and (5-4.9), where e is the total charge in the spatial region T.

Equations (5-4.1') and (5-4.4') have a simple physical significance. They represent the mathematical formulation of simple experimental facts and form the basis on which Maxwell's equations are derived. Thus Eq. (5-4.6') is the

generalization of the known Biot-Savart law. Equation (5-4.7′) expresses the Faraday induction law, and Eq. (5-4.9′) can be derived directly from Coulomb's law. Equation (5-4.8′) results from the fact that the lines of force of the magnetic field are closed.

If the medium is inhomogeneous, then other boundary conditions are added to Maxwell's equations. On the boundary surface between two different mediums (1) and (2), the following conditions must be satisfied:

$$E_s^{(1)} = E_s^{(2)} \quad \text{(continuity of the tangential components of the electrical field } \boldsymbol{E}\text{)}, \tag{5-4.14}$$

$$H_s^{(1)} = H_s^{(2)} \quad \text{(continuity of the tangential components of the magnetic field } \boldsymbol{H}\text{)}, \tag{5-4.15}$$

and further,

$$B_{n_1}^{(1)} = B_{n_2}^{(2)} \quad \text{(continuity of the normal component of the magnetic induction } \boldsymbol{B}\text{)} \tag{5-4.16}$$

$$D_{n_1}^{(1)} - D_{n_2}^{(2)} = 4\pi\nu \quad \text{or} \quad \varepsilon_1 E_{n_1}^{(1)} - \varepsilon_2 E_{n_2}^{(2)} = 4\pi\nu\,, \tag{5-4.17}$$

where $\boldsymbol{n}_1$ and $\boldsymbol{n}_2$ are the normals to the surfaces separating both mediums. Here $\boldsymbol{n}_1$ is directed into the interior of the first medium, and $\boldsymbol{n}_2$ is directed into the interior of the second medium; ν denotes the surface density of charge. We easily obtain these conditions from Eqs. (5-4.6′) to (5-4.9′).

From Maxwell's equations and the corresponding boundary conditions, the electromagnetic field can be uniquely calculated from a given initial condition. In other words, for the unique determination of the field the conditions, Eqs. (5-4.14) and (5-4.15) are sufficient (that is, the requirement that the tangential components of both field vectors $\boldsymbol{E}$ and $\boldsymbol{H}$ are continuous).

If we are dealing with a stationary electromagnetic field (that is, independent of the time), then Maxwell's equations assume the following form:

$$\begin{aligned} \operatorname{rot} \boldsymbol{E} &= 0\,, \\ \operatorname{div} \varepsilon \boldsymbol{E} &= 4\pi\sigma\,, \\ \operatorname{rot} \boldsymbol{H} &= \frac{4\pi}{c}\sigma\boldsymbol{E} + \frac{4\pi}{c}\boldsymbol{j}^{(e)}\,, \\ \operatorname{div} \mu \boldsymbol{H} &= 0\,. \end{aligned}$$

Moreover, if the medium is nonconducting, i.e., $\sigma = 0$, then there exist two systems of equations, independent of each other, for the electric and magnetic field:

$$\left.\begin{aligned} \operatorname{rot} \boldsymbol{E} &= 0\,, \\ \operatorname{div} \varepsilon \boldsymbol{E} &= 4\pi\rho\,, \end{aligned}\right\} \quad \text{equations of electrostatics;}$$

$$\left.\begin{aligned} \operatorname{rot} \boldsymbol{H} &= \frac{4\pi}{c}\boldsymbol{j}^{(e)}\,, \\ \operatorname{div} \mu \boldsymbol{H} &= 0\,, \end{aligned}\right\} \quad \text{equations of magnetostatics.}$$

The electrostatic equations were considered in Chapter 4 and in the applications of Chapter 4.

In the case of a homogeneous medium, we easily obtain an equation for the individual vectors $\boldsymbol{E}$ and $\boldsymbol{H}$. We therefore assume that $\rho = 0$ and $\boldsymbol{j}^{(e)} = 0$. By applying the curl operator to Eq. (5-4.6), we have

$$\text{rot rot } \boldsymbol{H} = \frac{\varepsilon}{c}\frac{\partial}{\partial t}\text{rot } \boldsymbol{E} + \frac{4\pi}{c}\sigma \text{ rot } \boldsymbol{E}$$

from which, because of Eq. (5-4.7) and

$$\text{rot rot } \boldsymbol{H} = \text{grad div } \boldsymbol{H} - \Delta \boldsymbol{H} ,$$

we obtain the relation

$$\text{grad div } \boldsymbol{H} \quad \Delta \boldsymbol{H} - -\frac{\varepsilon\mu}{c^2}\frac{\partial^2 \boldsymbol{H}}{\partial t^2} - \frac{4\pi\sigma\mu}{c^2}\frac{\partial \boldsymbol{H}}{\partial t} ,$$

or

$$\Delta \boldsymbol{H} - \frac{1}{a^2}\frac{\partial^2 \boldsymbol{H}}{\partial t^2} + \frac{4\pi\sigma\mu}{c^2}\frac{\partial \boldsymbol{H}}{\partial t} , \qquad \left(a^2 - \frac{c^2}{\varepsilon\mu}\right), \tag{5-4.18}$$

since

$$\text{div } \boldsymbol{H} = 0 .$$

In a corresponding manner, we obtain for $\boldsymbol{E}$ the equation

$$\Delta \boldsymbol{E} = \frac{1}{a^2}\frac{\partial^2 \boldsymbol{E}}{\partial t^2} + \frac{4\pi\sigma\mu}{c^2}\frac{\partial \boldsymbol{E}}{\partial t} . \tag{5-4.19}$$

In particular, the components E_x, E_y, E_z and H_x, H_y, H_z satisfy Eqs. (5-4.18) or (5-4.19), that is, each of them satisfies the equation

$$\Delta u = \frac{1}{a^2}\frac{\partial^2 u}{\partial t^2} + \frac{4\pi\sigma\mu}{c^2}\frac{\partial u}{\partial t} . \tag{5-4.20}$$

The type of this equation is determined by the properties of the medium and the character of the process. If the medium is nonconducting ($\sigma = 0$), we obtain the ordinary wave equation

$$\Delta u = \frac{1}{a^2}\frac{\partial^2 u}{\partial t^2} , \tag{5-4.21}$$

that is, in a nonconducting damp-free medium, the electromagnetic process propagates with the velocity $a = c/\sqrt{\varepsilon\mu}$. In particular, the propagation velocity in a vacuum is equal to the velocity of light c.

If the medium possesses a large conductivity and the displacement current in contrast to the conduction current can be neglected, then the process is described by the parabolic differential equation

$$\Delta u = \frac{4\pi\sigma\mu}{c^2}\frac{\partial u}{\partial t} . \tag{5-4.22}$$

In the general case in which the conduction current and the displacement current are of the same order of magnitude, Eq. (5-4.20) represents a hyperbolic differential equation. The propagation process is then connected with a damping which is produced by an energy loss because of the conductivity.

For example, for induced processes, in diffraction problems, we set

$$u = v(x, y, z)e^{i\omega t}$$

and arrive at the elliptic differential equation

$$\Delta v + (k^2 + iq^2)v = 0 , \tag{5-4.23}$$

where

$$k^2 = \frac{\omega^2}{a^2} , \qquad q^2 = \frac{4\pi\sigma\mu\omega}{c^2} . \tag{5-4.24}$$

Static fields, as already noted in Chapter 4, are represented by the Laplace differential equation.

2. *Potential of an electromagnetic field.* For the determination of an electromagnetic field, we must generally calculate six quantities, namely the components of the vectors $\boldsymbol{E}$ and $\boldsymbol{H}$. In a number of cases, the problem can be reduced to the calculation of four quantities (under certain circumstances even fewer). To this end, we introduce the vector potential $\boldsymbol{A}$ and the scalar potential φ of the field as follows: We consider Maxwell's equations in a homogeneous medium, say in a vacuum. From the equation

$$\operatorname{div} \boldsymbol{H} = 0 , \tag{5-4.8'}$$

it follows that the magnetic field vector $\boldsymbol{H}$ is source-free and, therefore, with the help of another field vector $\boldsymbol{A}$ can be represented in the form

$$\boldsymbol{H} = \operatorname{rot} \boldsymbol{A} . \tag{5-4.25}$$

By introducing this expression into

$$\operatorname{rot} \boldsymbol{E} = -\frac{1}{c}\frac{\partial \boldsymbol{H}}{\partial t} ,$$

we obtain

$$\operatorname{rot}\left[\boldsymbol{E} + \frac{1}{c}\frac{\partial \boldsymbol{A}}{\partial t}\right] = 0 ,$$

that is, the vector $\boldsymbol{E} + (1/c)(\partial \boldsymbol{A}/\partial t)$ possesses a potential. Thus this vector can be represented as the gradient of a scalar function φ:

$$\boldsymbol{E} + \frac{1}{c}\frac{\partial \boldsymbol{A}}{\partial t} = -\operatorname{grad} \varphi . \tag{5-4.26}$$

From this equation it follows that

$$\boldsymbol{E} = -\operatorname{grad} \varphi - \frac{1}{c}\frac{\partial \boldsymbol{A}}{\partial t} .$$

The vector potential $\boldsymbol{A}$ introduced in this manner and the scalar potential φ are not uniquely determined. From formulas (5-4.25) and (5-4.26) we know that we obtain one and the same field when we replace $\boldsymbol{A}$ and φ by the potentials

$$\boldsymbol{A}' = \boldsymbol{A} + \operatorname{grad} F , \qquad \varphi' = \varphi - \frac{1}{c}\frac{\partial F}{\partial t} ,$$

where F is an arbitrary function. In order to remove this ambiguity, we impose on $\boldsymbol{A}$ and φ that auxiliary condition

$$\operatorname{div} \boldsymbol{A} + \frac{1}{c}\frac{\partial \varphi}{\partial t} = 0 , \tag{5-4.27}$$

which is ordinarily known as the Lorentz condition. We shall now show that under this condition the potential $\boldsymbol{A}$ and φ satisfy the equations

$$\Delta\varphi - \frac{1}{c^2}\frac{\partial^2 \varphi}{\partial t^2} = -4\pi\rho , \tag{5-4.28}$$

$$\Delta \boldsymbol{A} - \frac{1}{c^2}\frac{\partial^2 \boldsymbol{A}}{\partial t^2} = -\frac{4\pi}{c}\boldsymbol{j} , \tag{5-4.29}$$

where ρ is the given charge density and $\boldsymbol{j}$ is the given current density.

If we put expressions (5-4.25) and (5-4.26) into Eq. (5-4.6), then by using the identity

$$\operatorname{rot} \operatorname{rot} \boldsymbol{A} = \operatorname{grad} \operatorname{div} \boldsymbol{A} - \Delta \boldsymbol{A}$$

we obtain the expression

$$\operatorname{grad}\left(\operatorname{div} \boldsymbol{A} + \frac{1}{c}\frac{\partial \varphi}{\partial t}\right) - \Delta \boldsymbol{A} = -\frac{1}{c^2}\frac{\partial^2 \boldsymbol{A}}{\partial t^2} + \frac{4\pi}{c}\boldsymbol{j} .$$

As a result of the above and condition (5-4.27), Eq. (5-4.29) follows. Now if we substitute expression (5-4.26) into the fourth Maxwell equation

$$\operatorname{div} \boldsymbol{E} = 4\pi\rho \tag{5-4.9'}$$

and use condition (5-4.27), we obtain Eq. (5-4.28) for φ.

In the case of a homogeneous conducting medium ($\sigma \neq 0$), the potential is introduced by means of the expression

$$\boldsymbol{B} = \operatorname{rot} \boldsymbol{A} , \qquad \boldsymbol{E} = -\operatorname{grad} \varphi - \frac{1}{c}\frac{\partial \boldsymbol{A}}{\partial t} . \tag{5-4.30}$$

$\boldsymbol{A}$ and φ are then related to each other by the expression

$$\operatorname{div} \boldsymbol{A} + \frac{\varepsilon\mu}{c}\frac{\partial \varphi}{\partial t} + \frac{4\pi\mu\sigma}{c}\varphi = 0 \tag{5-4.31}$$

and satisfy the equations

$$\Delta \boldsymbol{A} - \frac{\varepsilon\mu}{c^2}\frac{\partial^2 \boldsymbol{A}}{\partial t^2} - \frac{4\pi\mu\sigma}{c}\frac{\partial \boldsymbol{A}}{\partial t} = -\frac{4\pi\mu}{c}\boldsymbol{j}^{e} , \tag{5-4.32}$$

$$\Delta\varphi - \frac{\varepsilon\mu}{c^2}\frac{\partial^2 \varphi}{\partial t^2} - \frac{4\pi\mu\sigma}{c}\frac{\partial \varphi}{\partial t} = -\frac{4\pi\rho}{\varepsilon} . \tag{5-4.33}$$

For these equations, the considerations used above for Eqs. (5-4.28) and (5-4.29) are valid.

If no free charges are present ($\rho \equiv 0$), then $\varphi = 0$, and the field vectors can be expressed through the vector potential $\boldsymbol{A}$ alone, which here satisfies the auxiliary condition

$$\operatorname{div} \boldsymbol{A} = 0 .$$

In a series of cases, the electromagnetic field can be described by a vector field which has only one component different from zero.

Some of these examples will be considered in the following discussion. (see also Section 7-5.1.)

3. *The electromagnetic field of an oscillator.*

(a) In the theory of radiation of electromagnetic waves, the notion of an oscillator is often used. This concept is closely related to that of linear current. An oscillator represents a linear current of infinitesimally small length.

We shall consider a rectilinear current L, whose strength undergoes time dependent changes. In the simplest case, we shall assume that the strength of the current does not change with the length of the conductor.

One such current, constant along the conductor, is related to the presence of charges at the ends of the conductor which changes with time. By analogy with an electrostatic dipole, which we represent by a system of two charges e and $-e$, an oscillator can be characterized by its moment

$$\boldsymbol{p}(t) = e(t)\boldsymbol{l} \,. \tag{5-4.34}$$

The strength of the current in the oscillator is then obviously equal to

$$J(t) = e(t) \,,$$

so that

$$\frac{d\boldsymbol{p}}{dt} = J(t)\boldsymbol{l} \,. \tag{5-4.35}$$

The product $J(t)\boldsymbol{l} = \boldsymbol{J}_0(t)$ is called the moment of the current.

(b) We seek the electromagnetic field which is generated by an oscillator with the moment

$$\boldsymbol{p} = \boldsymbol{p}_0 f(t) \tag{5-4.36}$$

in unbounded space under the assumption that $\sigma = 0$, $\varepsilon = 1$, $\mu = 1$ (vacuum).

Moreover, we consider the generation of an electromagnetic field by a rectilinear current L, when we regard the oscillator as the limiting case.

Outside of the current L, the electromagnetic field is defined by Maxwell's equations

$$\begin{aligned} \operatorname{rot} \boldsymbol{H} &= \frac{1}{c}\frac{\partial \boldsymbol{E}}{\partial t} \,, & \operatorname{div} \boldsymbol{H} &= 0 \,, \\ \operatorname{rot} \boldsymbol{E} &= -\frac{1}{c}\frac{\partial \boldsymbol{H}}{\partial t} \,, & \operatorname{div} \boldsymbol{E} &= 0 \,. \end{aligned} \tag{5-4.37}$$

On the line of current L, the field vectors $\boldsymbol{H}$ and $\boldsymbol{E}$ must have a singularity which, moreover, is characterized by the fact that the circulation about one of the current lines enclosed by an infinitesimally small circle K_ε has the value

$$\lim_{\varepsilon \to 0} \oint_{K_\varepsilon} H_S \, ds = \frac{4\pi}{c} J \,, \tag{5-4.38}$$

where $J = J(t)$ denotes the strength of the current.

From this condition it follows that the component H_S must have a singularity of the type

$$H_S \approx \frac{2J}{c\rho} \tag{5-4.39}$$

on the line of current. Here, $\rho = \overline{M_0P}$, M_0 is a point on L and P is a point on K_ε ($\rho = \varepsilon$).

For the unique determination of the field, initial conditions are given which we assume to be homogeneous.

For the solution of the problem, it is convenient to use the potentials $\boldsymbol{A}$ and φ, introduced above, with whose help the field can be described as follows:

$$\begin{aligned} \boldsymbol{H} &= \operatorname{rot} \boldsymbol{A} \\ \boldsymbol{E} &= -\frac{1}{c}\frac{\partial \boldsymbol{A}}{\partial t} - \operatorname{grad} \varphi \,, \end{aligned} \tag{5-4.40}$$

where

$$\operatorname{div} \boldsymbol{A} + \frac{1}{c}\frac{\partial \varphi}{\partial t} = 0 \,, \tag{5-4.41}$$

The vector potential satisfies, outside of L, the homogeneous wave equation

$$\Delta \boldsymbol{A} - \frac{1}{c^2}\frac{\partial^2 \boldsymbol{A}}{\partial t^2} = 0 \,. \tag{5-4.42}$$

We now introduce a rectangular system of coordinates, whose z-axis has the direction of L. Then we set

$$\boldsymbol{A} = A(x, y, z)\boldsymbol{z}^0 \,,$$

where $\boldsymbol{z}^0$ is the unit vector in the direction of the z-axis.

Obviously, the function $A(x, y, z)$ outside of the line current L, satisfies the homogeneous wave equation

$$\Delta A - \frac{1}{c^2}\frac{\partial^2 A}{\partial t^2} = 0 \,. \tag{5-4.43}$$

For the determination of the singularity of $\boldsymbol{A}$ on the line current, we use condition (5-4.39). From (5-4.40) there follows

$$H_S = \frac{\partial A}{\partial \rho} \,.$$

By use of condition (5-4.39), we then find that the function $A(x, y, z)$ at the points of the line current L must have a singularity of the form

$$A \approx \frac{2J}{c} \ln \rho \,. \tag{5-4.44}$$

The potential A is taken in the form

$$A(P, t) = \int_L A_1(P, M, t)\, dl_M \qquad (P = P(x, y, z)) \,. \tag{5-4.45}$$

Therefore, $\boldsymbol{A}_0(P, M, t) = A_1(P, M, t)\, d\boldsymbol{l}_M$ is the vector potential of the elementary current of the oscillator whose moment is equal to $J_0 = J\, dl$.

Hence, if the potential A possesses the required singularity, then $A_0(P, M, t)$ must have a singularity of the form

$$A_0(P, M, t) \approx \frac{J_0(t)}{cr_{PM}} .$$

Namely, if we assume that A_0 has the desired singularity, and calculate the value of A in the neighborhood of L $(0 < z < l)$ according to formula (5-4.45), then there results

$$A \approx J\int_0^l \frac{1}{cr_{MP}}\,d\zeta = \frac{J}{c}\int_0^l \frac{d\zeta}{\sqrt{\rho^2 + (z-\zeta)^2}}$$

$$= \frac{J}{c}\ln\left[z - \zeta + \sqrt{\rho^2 + (z-\zeta)^2}\right]_0^l = \frac{J}{c}\ln\frac{z - l + \sqrt{\rho^2 + (z-l)^2}}{z + \sqrt{\rho^2 + z^2}}$$

$$= \frac{J}{c}\ln\frac{(l-z)\left[-1 + 1 + \dfrac{\rho^2}{2(l-z)^2} + \cdots\right]}{z + \sqrt{\rho^2 + z^2}} = \frac{2J}{c}\ln\rho + \cdots .$$

Here, the dots imply terms which, for $\rho = 0$, assume a finite value.

(c) Consequently, with respect to the variables $P(x, y, z)$ and t, $A_0(P, M, t)$ must everywhere satisfy the wave equation

$$\Delta A_0 - \frac{1}{c^2}\frac{\partial^2 A_0}{\partial t^2} = 0 \tag{5-4.46}$$

with the exception of the point M, while at this point it possesses a singularity of the form

$$A_0 \approx \frac{J_0(t)}{cr_{MP}} . \tag{5-4.47}$$

The initial conditions, according to previous agreement, are homogeneous.

The solution of this problem, as we have seen in Chapter 5, is given by the retarded potential

$$A_0(M, P, t) = \frac{J_0\left(t - \dfrac{r_{MP}}{c}\right)}{cr_{MP}} . \tag{5-4.48}$$

As we have noted above, the moment of current is equal to

$$\boldsymbol{J}(t) = J(t)\,d\boldsymbol{l} = \frac{d\boldsymbol{p}}{dt} = \boldsymbol{p}_0 \mathring{f}(t) . \tag{5-4.49}$$

Therefore, the field vector of the oscillator can be represented in the form

$$\boldsymbol{A}_1 = \frac{\boldsymbol{p}_0 \mathring{f}\left(t - \dfrac{r}{c}\right)}{cr} . \tag{5-4.50}$$

Often, instead of the vector potential, the polarization potential (the Hertz vector) $\boldsymbol{\pi}$ is used and we obtain

$$\boldsymbol{A} = \frac{1}{c}\frac{\partial \boldsymbol{\pi}}{\partial t} . \tag{5-4.51}$$

The Hertz vector π likewise satisfies the homogeneous wave equation

$$\Delta\pi - \frac{1}{c^2}\frac{\partial^2\pi}{\partial t^2} = 0 \tag{5-4.52}$$

and is connected with the scalar potential by the relation

$$\varphi = -\operatorname{div}\pi\ . \tag{5-4.53}$$

The field vectors $\boldsymbol{E}$ and $\boldsymbol{H}$ can be expressed through the polarization potential as follows:

$$\boldsymbol{E} = \operatorname{grad}\operatorname{div}\pi - \frac{1}{c^2}\frac{\partial^2\pi}{\partial t^2} = \operatorname{rot}\operatorname{rot}\pi \tag{5-4.54}$$

$$\boldsymbol{H} = \frac{1}{c}\operatorname{rot}\frac{\partial\pi}{\partial t}\ . \tag{5-4.55}$$

By using Eq. (5-4.50), we obtain for the polarization potential of the oscillator the expression

$$\pi_0 = \frac{\boldsymbol{p}_0 f\left(t - \frac{r}{c}\right)}{r}$$

or

$$\pi_0 = \frac{\boldsymbol{p}\left(t - \frac{r}{c}\right)}{r}\ . \tag{5-4.56}$$

(d) For the calculation of the electrical field $\boldsymbol{E}$ and the magnetic field $\boldsymbol{H}$, we use spherical coordinates r, ϑ, φ, at whose origin lies the oscillator. The z-axis ($\vartheta = 0$) has the direction of the vector $\boldsymbol{p}_0$ (Figure 82). We denote

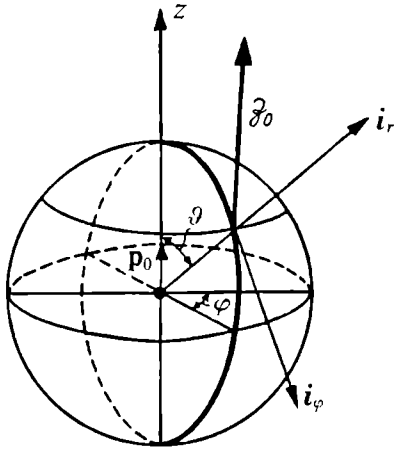


FIG. 82

the unit vectors of the polar coordinate system, by $\boldsymbol{i}_r, \boldsymbol{i}_\vartheta, \boldsymbol{i}_\varphi$.

The vector π_0 parallel to the vector $\boldsymbol{p}$ can be represented in the form

$$\begin{aligned} \pi_0 &= z_0\cos\vartheta\boldsymbol{i}_r - z_0\sin\vartheta\boldsymbol{i}_\vartheta\ , \\ z_0 &= |\pi_0|\ . \end{aligned} \tag{5-4.57}$$

By substituting (5-4.57) in Eqs. (5-4.54) and (5-4.55) and using the representation of rot $\boldsymbol{F}$ in spherical coordinates

$$\operatorname{rot} \boldsymbol{F} = \frac{1}{r \sin \vartheta}\left[\frac{\partial}{\partial \vartheta}(\sin \vartheta F_\varphi) - \frac{\partial F_\vartheta}{\partial \varphi}\right]\boldsymbol{i}_r - \frac{1}{r}\left[\frac{1}{\sin \vartheta}\frac{\partial F_r}{\partial \varphi} - \frac{\partial}{\partial r}(rF_\varphi)\right]\boldsymbol{i}_\vartheta + \frac{1}{r}\left[\frac{\partial}{\partial r}(rF_\vartheta) - \frac{\partial F_r}{\partial \vartheta}\right]\boldsymbol{i}_\varphi$$

and the relationship $z_0 = z_0(r, t)$, we obtain the expressions

$$E_r = -\frac{2\cos\vartheta}{r}\frac{\partial Z_0}{\partial r}, \qquad E_\vartheta = \frac{\sin\vartheta}{r}\frac{\partial}{\partial r}\left(r\frac{\partial Z_0}{\partial r}\right), \qquad E_\varphi = 0, \tag{5-4.58}$$

$$H_r = 0, \qquad H_\vartheta = 0, \qquad H_\varphi = -\sin\vartheta\frac{1}{c}\frac{\partial^2 Z_0}{\partial r\,\partial t}. \tag{5-4.59}$$

From Eqs. (5-4.58) and (5-4.59) it follows that the electric and magnetic fields of the oscillator are perpendicular to each other.

(e) We shall now consider the case in which the dipole moment depends periodically on the time

$$\boldsymbol{p}(t) = \boldsymbol{p}_0 e^{-i\omega t}.$$

In this case formulas (5-4.58) and (5-4.59) read

$$E_r = 2\cos\vartheta\left(\frac{1}{r^2} - \frac{ik}{r}\right)Z_0, \qquad E_\vartheta = \sin\vartheta\left(\frac{1}{r^2} - \frac{ik}{r} - k^2\right)Z_0 \qquad \left(k = \frac{\omega}{c}\right), \tag{5-4.60}$$

$$H_\varphi = ik\sin\vartheta\left(ik - \frac{1}{r}\right)Z_0,$$

where

$$Z_0 = p_0\frac{e^{ikr}}{r}e^{-i\omega t}. \tag{5-4.61}$$

If we proceed from formula (5-4.60), we can easily establish the particular characteristics of the oscillator field. At distances which are small compared to the wave length $\lambda = 2\pi/k$ ($kr \ll 1$), we can limit ourselves to one term in formula (5-4.60). The formula obtained in this manner for the electric field corresponds to the field of a stationary dipole, whose electric moment p is equal to the instantaneous value of the oscillator moment $p(t)$. For the magnetic field we obtain an expression which corresponds to the Biot-Savart law. At larger distances from the dipole, $R \gg \lambda$ ($kr \gg 1$), all terms in formula (5-4.60) which are of higher order than $1/r$ can be neglected. Hence we obtain

$$E_r = 0, \qquad E_\vartheta = H_\varphi = -k^2\sin\vartheta Z_0; \tag{5-4.62}$$

that is, the field is transverse with respect to the direction of propagation. Such a subdomain of the field in which the radiation field is transverse is called a wave zone of the oscillator. For determination of the energy flow through a spherical surface of radius R whose center is at the oscillator, we must first calculate the Poynting vector

$$\boldsymbol{S} = \frac{c}{4\pi}|[\boldsymbol{EH}]| = \frac{c}{4\pi} EH$$

and integrate this over the entire sphere.

From formulas (5-4.61) and (5-4.62) it follows that in a wave zone the real part of the vector H_φ and E_ϑ is defined by the expression

$$H_\varphi = E_\vartheta = -\frac{\omega^2 \sin\vartheta}{c^2 r} p_0 \cos\omega\left(t - \frac{r}{c}\right).$$

Hence, we obtain

$$S = \frac{p_0^2\omega^4}{4\pi} \frac{\sin^2\vartheta}{R^2 c^3} \cos^2\omega\left(t - \frac{r}{c}\right).$$

Accordingly, the energy flow through a sphere of radius R during an entire period $T = 2\pi/\omega$ is equal to

$$\mathcal{E} = \frac{1}{T}\int_0^T dt \int_0^\pi \int_0^{2\pi} SR^2 \sin\vartheta \, d\vartheta \, d\varphi = \frac{2p_0^2\omega^4}{3c^3} \frac{1}{T}\int_0^T \cos^2\omega\left(t - \frac{r}{c}\right) dt$$

or

$$\mathcal{E} = \frac{p_0^2\omega^4}{3c^3}.$$

The energy emitted by a harmonic oscillator is, therefore, proportional to the fourth power of the frequency

$$\mathcal{E} \sim \omega^4 \quad \text{or} \quad \mathcal{E} \sim \frac{1}{\lambda^4},$$

where λ is the wave length.

6

SPATIAL HEAT PROPAGATION

6-1. HEAT PROPAGATION IN UNBOUNDED SPACE

1. Green's function

In Chapter 3 the propagation of heat in an homogeneous isotropic media was described by the heat-conduction equation

$$u_t = a^2 \Delta u , \qquad \left(a^2 = \frac{k}{c\rho}\right) . \tag{6-1.1}$$

Here $u(M, t)$ is the temperature at point $M(x, y, z)$ at time t, ρ is the density, c is the specific heat, $k = \text{const.}$ is the heat conductivity, and $a^2 = k/c\rho$ is the temperature conductivity. Equation (6-1.1) is also known as the diffusion equation. In this case u denotes the concentration of the diffusing substance and $a^2 = D$ is the diffusion coefficient.

We begin with the consideration of the following problem in an unbounded space:

Determine the solution of the inhomogeneous heat conduction equation

$$u_t = a^2 \Delta u + \frac{f}{c\rho} \tag{6-1.2}$$

(f denotes the density of the heat source) which satisfies the initial condition

$$u(x, y, z, 0) = \varphi(x, y, z) . \tag{6-1.3}$$

The solution of this problem can be represented in the form

$$u = u_1 + u_2 ,$$

where u_1 is the solution of the homogeneous equation (6-1.1) with inhomogeneous initial conditions, and u_2 is the solution of the inhomogeneous equation (6-1.2) with homogeneous initial conditions. For the investigation of the corresponding homogeneous problem, we see that it is sufficient to determine Green's function for its solution.

We shall now construct Green's function for the heat conduction equation in an unbounded space.

First we prove the following lemma, which will be of use to us later:

Lemma. If a solution of the equation $\Delta u - (1/a^2)u_t = 0$ depends only on r and t, then the function $v = ru$ satisfies the equation

$$\frac{\partial^2 v}{\partial r^2} = \frac{1}{a^2}\frac{\partial v}{\partial t} . \tag{6-1.4}$$

Thus, if we write the Laplace differential expression in polar coordinates, we see that the function $u = u(r, t)$ satisfies the equation

$$\frac{\partial^2 u}{\partial r^2} + \frac{2}{r}\frac{\partial u}{\partial r} - \frac{1}{a^2}\frac{\partial u}{\partial t} = 0 \quad \text{or} \quad \frac{1}{r}\frac{\partial^2 (ru)}{\partial r^2} - \frac{1}{a^2}\frac{\partial u}{\partial t} = 0 .$$

Now if we set $ru = v$, then for v we obtain equation (6-1.4).[13]

At the origin of coordinates lies a continuously acting heat source with a constant intensity q, while in the remaining space the initial temperature is equal to 0:

$$u(r, 0) = 0 \quad \text{for } r \neq 0 .$$

Obviously, in this case the temperature u is a function which depends only on r and t.

The presence of the heat source for $r = 0$ indicates that the heat flow in a unit time through a sphere S_ε (with the center at $r = 0$ and radius ε) as $\varepsilon \to 0$ is equal to q, that is,

$$\lim_{\varepsilon \to 0} \left[-\int_{S_\varepsilon} \int k \frac{\partial u}{\partial n} d\sigma \right] = q .$$

Since the normal derivative $\partial u/\partial n = \partial u/\partial r$ is constant because of symmetry on the spherical surface S_ε, we have

$$-k\frac{\partial u}{\partial r} \cdot 4\pi r^2 \Big|_{r=\varepsilon} \to q \quad \text{for } \varepsilon \to 0 .$$

This means that the derivative $\partial u/\partial r$ for $r = 0$ possesses a singularity of the form $-q/4\pi k r^2$. Consequently, for $r = 0$, the function itself must possess a singularity of the form

$$u \sim \frac{q}{4\pi k r} ,$$

so that the product $ru = v$ for $r = 0$ remains bounded.

The function v defined by the conditions

$$\frac{\partial^2 v}{\partial r^2} = \frac{1}{a^2}\frac{\partial v}{\partial t} ,$$

$$v(0, t) = \frac{q}{4\pi k} = v_0 , \tag{6-1.5}$$

$$v(r, 0) = 0$$

can be represented by

$$v(r, t) = v_0 \left[1 - \Phi\left(\frac{r}{2\sqrt{a^2 t}} \right) \right] = q \frac{1}{4\pi k} \frac{2}{\sqrt{\pi}} \int_{r/(2\sqrt{a^2 t})}^{\infty} e^{-\alpha^2} d\alpha$$

[13] See Section 5-1.1.

[see formula (3 3.35)]. Consequently, the heat-conduction equation for a continuously acting source of intensity q, which lies at the origin ($r = 0$), has a solution given by

$$u(r, t) = qU(r, t) = q\frac{1}{4\pi k}\frac{1}{r}\frac{2}{\sqrt{\pi}}\int_{r/(2\sqrt{a^2t})}^{\infty} e^{-\alpha^2}d\alpha , \qquad (6\text{-}1.6)$$

where $U(r, t)$ is the temperature corresponding to a unit source ($q = 1$).

Now, in order to go to the case of an instantaneously acting source, we consider a source of intensity q lying at the point (ξ, η, ζ) which may act continuously during a time interval τ. Such a source is equivalent to two sources of intensities, $+q$ and $-q$, one of which acts at time $t = 0$, the other at time $t = \tau$. Thus the temperature distribution can be described by

$$u_\tau(r, t) = q[U(r, t) - U(r, t - \tau)] .$$

During the time interval τ, the amount of heat $Q = q\tau$ is set free, so that we obtain

$$u_\tau(r, t) = \frac{Q}{\tau}[U(r, t) - U(r, t - \tau)] .$$

If we now pass to the limit as $\tau \to 0$ and assume that Q remains constant, we obtain

$$u_0(r, t) = \lim_{\tau \to 0} u_\tau(r, t) = Q\frac{\partial U}{\partial t} = \frac{Q}{4\pi k r}\frac{2}{\sqrt{\pi}}e^{-r^2/4a^2t}\frac{r}{4\sqrt{a^2t^3}}a^2$$

or

$$u_0(r, t) = \frac{Q}{c\rho}G(x, y, z, t; \xi, \eta, \zeta) ,$$

where

$$G(x, y, z, t; \xi, \eta, \zeta) = \left(\frac{1}{2\sqrt{\pi a^2 t}}\right)^3 e^{-[(x-\xi)^2 + (y-\eta)^2 + (z-\zeta)^2]/4a^2t} . \qquad (6\text{-}1.7)$$

is Green's function of the heat source. It represents the temperature at the point (x, y, z) at time t, which is produced by the point source of intensity $Q = c\rho$ at the point (ξ, η, ζ), at the time $t = 0$.

It can easily be shown that

$$\int_{-\infty}^{\infty}\int\int G(x, y, z, t; \xi, \eta, \zeta)\,d\xi d\eta d\zeta = 1 . \qquad (6\text{-}1.8)$$

The triple integral (6-1.8) can thus be written as a product of three integrals, each of which is equal to unity:

$$\int_{-\infty}^{\infty}\frac{1}{2\sqrt{\pi a^2 t}}e^{-(x-\xi)^2/4a^2t}d\xi = \frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty}e^{-\alpha^2}d\alpha = 1 , \quad \left(\alpha = \frac{\xi - x}{2\sqrt{a^2t}}\right) .$$

From Eq. (6-1.7) it is apparent that Green's function G is symmetric:

$$G(x, y, z, t; \xi, \eta, \zeta) = G(\xi, \eta, \zeta, t; x, y, z) .$$

This property is the mathematical expression of the reciprocity principle: The action, due to a source lying at the point (ξ, η, ζ), at the point (x, y, z) is equal to the action at the point (ξ, η, ζ), due to the same source found at the point (x, y, z). For the variable t such a symmetry is not valid. This corresponds to the irreversibility of heat processes with respect to time.

In order to determine Green's function G for two variables, we consider a straight line, parallel to the z-axis and passing through the point (ξ, η). On this straight line let there be distributed an infinitely long linear source; we denote by $Q = \text{const}$ the intensity of the source per unit of length. Green's function G_2 of such a source does not depend on z. It is determined completely by its value in the x, y-plane. For the calculation of G_2 we therefore proceed from the fact that the amount of heat leaving the element $d\zeta$ is

$$dQ = \bar{Q}\, d\zeta .$$

Then the temperature distribution in space is given by the integral

$$\bar{u} = \int_{-\infty}^{\infty} \frac{\bar{Q}\, d\zeta}{c\rho} G(x, y, z, t; \xi, \eta, \zeta) .$$

From the calculation of the integral

$$\int_{-\infty}^{\infty} e^{-(z-\zeta)^2/4a^2t}\, d\zeta = 2\sqrt{a^2t} \int_{-\infty}^{\infty} e^{-\alpha^2}\, d\alpha = 2\sqrt{\pi a^2 t} , \qquad \left(\alpha = \frac{\zeta - z}{2\sqrt{a^2t}}\right) ,$$

we obtain

$$\bar{u} = \frac{\bar{Q}}{c\rho} G_2 ,$$

where

$$G_2(x, y, t; \xi, \eta) = \left(\frac{1}{2\sqrt{\pi a^2 t}}\right)^2 e^{-[(x-\xi)^2 + (y-\eta)^2]/4a^2t} . \tag{6-1.8'}$$

By comparing (6-1.8′) with formula (6-1.7) we recognize the corresponding structure of Green's function for both two and three variables.

Green's function for the one-dimensional case can be obtained in the same manner. If we consider an infinite plane source with constant density Q, we obtain

$$\bar{\bar{u}} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{\bar{\bar{Q}}\, d\eta\, d\zeta}{c\rho} G(x, y, z, t; \xi, \eta, \zeta) = \frac{\bar{\bar{Q}}}{c\rho} \frac{1}{2\sqrt{\pi a^2 t}} e^{-(x-\xi)^2/4a^2t}$$
$$= \frac{\bar{\bar{Q}}}{c\rho} G_1(x, t, \xi) ,$$

where

$$G_1(x, t; \xi) = \frac{1}{2\sqrt{\pi a^2 t}} e^{-(x-\xi)^2/4a^2t}$$

is Green's function (source function) for one variable. In Chapter 3, we gave a graphical representation of $G(x, t; \xi)$. The properties of Green's function described there hold also for the spatial case.

2. Heat propagation in an unbounded space

We shall now use Green's function constructed in the preceding section to investigate the propagation of an initial temperature in an unbounded space. Also we shall find a solution of the equation

$$u_t = a^2 \Delta u \,, \tag{6-1.1}$$

which satisfies the initial condition

$$u(x, y, z, 0) = \varphi(x, y, z) \,. \tag{6-1.3}$$

The initial temperature state obviously can be represented as the superposition of the actions of the instantaneous sources which produce the initial temperature. Therefore, we consider an element of volume $d\xi\, d\eta\, d\zeta$, which contains the point (ξ, η, ζ). For the production of the initial temperature $\varphi(\xi, \eta, \zeta)$, the amount of heat $dQ = c\rho\varphi(\xi, \eta, \zeta)\, d\xi\, d\eta\, d\zeta$ must then be present in $d\xi\, d\eta\, d\zeta$.

At another point (x, y, z) at time t this amount of heat produces the temperature

$$\frac{dQ}{c\rho} G(x, y, z, t; \xi, \eta, \zeta) = G(x, y, z, t; \xi, \eta, \zeta)\varphi(\xi, \eta, \zeta)\, d\xi\, d\eta\, d\zeta \,. \tag{6-1.9}$$

On the basis of the superposition principle, the solution of this problem can be obtained by integration of Eq. (6-1.9) over the entire space:

$$u(x, y, z, t) = \iiint_{-\infty}^{\infty} G(x, y, z, t; \xi, \eta, \zeta)\varphi(\xi, \eta, \zeta)\, d\xi\, d\eta\, d\zeta \,. \tag{6-1.10}$$

We obtained formula (6-1.10) by heuristic considerations, without regard to the limits of its applicability; hence it is not a proof.

We shall now prove the following theorem:

Theorem. If φ is a piecewise continuous and bounded function, $|\varphi| < A$, then the function u defined by the relation

$$u(x, y, z, t) = \left(\frac{1}{2\sqrt{\pi a^2 t}}\right)^3 \iiint_{-\infty}^{\infty} e^{-[(x-\xi)^2 + (y-\eta)^2 + (z-\zeta)^2]/4a^2 t} \varphi(\xi, \eta, \zeta)\, d\xi\, d\eta\, d\zeta \tag{6-1.10'}$$

is

1. bounded in the entire space ($|u| < A$);
2. a solution of the heat conduction equation for $t > 0$;
3. for $t = 0$ continuous at the points of continuity of the function φ for which the condition $u(x, y, z, 0) = \varphi(x, y, z)$ holds.

For proof of 2 we use the following well-known lemma (see Section 3-3).

Lemma. If for an arbitrary value of the parameter ξ, $U(x, y, z, t; \xi)$ is a solution of the equation $\boldsymbol{L}(u) = 0$ ($\boldsymbol{L}$ denotes a linear differential operator), then the function

$$u(x, y, z, t) = \int U(x, y, z, t; \xi)\varphi(\xi)\, d\xi$$

is likewise a solution of the equation $L(u) = 0$, if the derivatives of the function u occurring in $L(u)$ can be calculated by differentiation under the integral sign.

In our case $U = G$ satisfies the heat-conduction equation for arbitrary ξ, η, ζ. As is well known, an improper integral can be differentiated with respect to a parameter under the integral sign, when (a) the derivative of the integrand is continuous with respect to the parameter, and (b) when the integral which we obtain after the formal differentiation converges uniformly.

If we formally differentiate the integral in formula (6-1.10′) with respect to x, then we obtain

$$\iiint_{-\infty}^{\infty} \left(-\frac{x-\xi}{2a^2t}\right) e^{-[(x-\xi)^2+(y-\eta)^2+(z-\zeta)^2]/4a^2t} \varphi(\xi, \eta, \zeta)\, d\xi\, d\eta\, d\zeta\,. \tag{6-1.11}$$

The integrand for $\bar{t} > t > 0$ is continuous, while the $\exp -([(x-\xi)^2+(y-\eta)^2+(z-\zeta)^2]/4a^2t)$ guarantees the uniform convergence, provided φ is bounded: $|\varphi| < A$. Corresponding results are obtained by repeated differentiation with respect to x and by differentiation with respect to t. This also holds true for differentiation with respect to y and z. Thus for $t > 0$, G satisfies all assumptions of the lemma and thereby satisfies the heat-conduction equation.

The boundedness of the function u defined by formula (6-1.10′), which we can also write in the form

$$u(M, t) = \iiint_{-\infty}^{\infty} G(M, M', t)\varphi(M')\, d\tau_{M'}$$
$$(M = M(x, y, z)\,, \qquad M' = M'(\xi, \eta, \zeta))\,,$$

can be determined directly if we use Eq. (6-1.8):

$$|u| < A \iiint_{-\infty}^{\infty} G\, d\tau = A\,. \tag{6-1.12}$$

We now proceed to the proof of the continuity of $u(x, y, z, t)$ for $t = 0$. For this purpose we consider a point $M_0(x_0, y_0, z_0)$, at which the function φ is continuous and prove that for each $\varepsilon > 0$, a $\delta(\varepsilon) > 0$ exists such that

$$|u(M, t) - \varphi(M_0)| < \varepsilon \quad \text{for} \quad |MM_0| < \delta(\varepsilon) \text{ and } t < \delta(\varepsilon)\,. \tag{6-1.13}$$

Now let T_1, containing the point M_0, be an auxiliary region whose size will be prescribed later. The remaining part of the space is denoted by T_2. By considerating the equations

$$u(M, t) = \iiint_{T_1} G(M, M', t)\varphi(M')\, d\tau_{M'} + \iiint_{T_2} G(M, M', t)\varphi(M')\, d\tau_{M'}\,,$$
$$\varphi(M_0) = \iiint_{T_1} G(M, M', t)\varphi(M_0)\, d\tau_{M'} + \varphi(M_0) \iiint_{T_2} G(M, M', t)\, d\tau_{M'}$$

and the relation $G(M, M', t) > 0$, we obtain

$$|u(M, t) - \varphi(M_0)| \leqq J_1 + J_2\,, \tag{6-1.14}$$

where

$$J_1 = \iint_{T_1} \int G(M, M', t)|\varphi(M') - \varphi(M_0)|\, d\tau_{M'} , \tag{6-1.15}$$

$$J_2 = 2A \iint_{T_2} \int G(M, M', t)\, d\tau_{M'} . \tag{6-1.16}$$

On the basis of the continuity of φ at the fixed point M_0 for an arbitrarily chosen $\eta > 0$, there exists a $\delta'(\eta) > 0$ such that

$$|\varphi(M') - \varphi(M_0)| < \eta , \quad \text{when } |M'M_0| < \delta'(\eta) .$$

Thus if the diameter of T_1 is not larger than $\delta'(\varepsilon/3)$, then

$$J_1 < \frac{\varepsilon}{3} \iint_{T_1} \int G\, d\tau_{M'} < \frac{\varepsilon}{3} \iiint_{-\infty}^{\infty} G\, d\tau = \frac{\varepsilon}{3} . \tag{6-1.17}$$

We shall now study more precisely the selection of the region T_1. For T_1, we can choose a sphere with center $M(x, y, z)$. This choice proves to be suitable for the estimation of the integral J_2. The estimation (6-1.17) of the integral J_1 remains valid when the radius ρ_0 of this sphere is chosen equal to

$$\tfrac{1}{2}\delta'\left(\frac{\varepsilon}{3}\right)$$

and $|MM_0| < \rho_0$.

By transforming to polar coordinates whose origin is at the point M, we obtain

$$\iint_{T_1} \int G\, d\tau = \left(\frac{1}{2\sqrt{\pi a^2 t}}\right)^3 \int_0^{\rho_0} e^{-r^2/4a^2t} r^2\, dr$$

$$= \frac{4}{\sqrt{\pi}} \int_0^{\rho_0/2\sqrt{a^2t}} \alpha^2 e^{-\alpha^2}\, d\alpha \xrightarrow[t \to 0]{} \frac{4}{\sqrt{\pi}} \int_0^{\infty} \alpha^2 e^{-\alpha^2}\, d\alpha = 1 ,$$

$$\left(\alpha = \frac{r}{2\sqrt{a^2 t}}\right) ,$$

since

$$\int_0^{\infty} \alpha^2 e^{-\alpha^2}\, d\alpha = -\tfrac{1}{2}\alpha e^{-\alpha^2}\Big|_0^{\infty} + \tfrac{1}{2}\int_0^{\infty} e^{-\alpha^2}\, d\alpha = \frac{\sqrt{\pi}}{4} .$$

Therefore we arrive at the relation

$$\iint_{T_2} \int G\, d\tau = 1 - \iint_{T_1} \int G\, d\tau \to 0 \quad \text{for } t \to 0 ,$$

that is, for each $\varepsilon > 0$, a $\delta''(\varepsilon)$ can be found such that

$$\iint_{T_2} \int G\, d\tau < \frac{\varepsilon}{3A} ,$$

and consequently

$$J_2 < \frac{2\varepsilon}{3} , \tag{6-1.18}$$

when $t < \delta''(\varepsilon)$.

Now we denote the minimum of the numbers $(1/2)\delta'(\varepsilon/3)$ and $\delta''(\varepsilon)$ by $\delta(\varepsilon)$. Then the inequality

$$|u(M, t) - \varphi(M_0)| < \varepsilon \quad \text{for } |MM_0| < \delta(\varepsilon) \quad \text{and } t < \delta(\varepsilon) \tag{6-1.13}$$

holds. Thus we have proved the continuity of $u(M, t)$ for $t = 0$ at each point M_0, where φ is continuous.

We shall now solve the inhomogeneous equation

$$u_t = a^2 \Delta u + \frac{f}{c\rho}$$

with the homogeneous initial condition

$$u(x, y, z, 0) = 0 .$$

For this purpose we consider a point (ξ, η, ζ) at the time $\tau < t$. The amount of heat which appears in the volume element $d\xi\, d\eta\, d\zeta$ during the time $d\tau$ and equals

$$dQ = f\, d\xi\, d\eta\, d\zeta\, d\tau$$

produces the temperature

$$\frac{1}{c\rho} G(x, y, z, t; \xi, \eta, \zeta, \tau) f(\xi, \eta, \zeta, \tau)\, d\xi\, d\eta\, d\zeta\, d\tau$$

at the point (x, y, z) at time t.

On the basis of the superposition principle, the solution of our problem can be written in the form

$$u(x, y, z, t) = \int_0^t \int\int\int_{-\infty}^{\infty} \frac{1}{c\rho} G(x, y, z, t; \xi, \eta, \zeta, \tau) f(\xi, \eta, \zeta, \tau)\, d\xi\, d\eta\, d\zeta\, d\tau . \tag{6-1.19}$$

We shall neither delve further into the proof of the correctness of this formula nor go into the conditions which guarantee its applicability.

The first and second boundary-value problems with homogeneous boundary conditions for the half space are solved by the reflection method.

6-2. HEAT PROPAGATION IN BOUNDED REGIONS

1. Scheme of the method of separation of variables

In Section 6-1 we investigated heat propagation in an unbounded space. To investigate heat propagation in a bounded region of space, certain boundary conditions must be added to the equation and to the corresponding initial condition. The simplest of these cases are those with boundary conditions of the first, second, or third kind.

To begin we formulate the simplest problem with a homogeneous boundary condition of the first kind:

Determine the solution of the heat-conduction equation

$$u_t = a^2 \Delta u \quad \text{in the interior of } T,$$

$$\text{with the initial condition } u(x, y, z, 0) \quad \varphi(x, y, z) , \tag{6-2.1}$$

and the boundary condition $u_\Sigma = 0$,

where Σ denotes the boundary of the region T.

Again, the solution of this problem begins with the help of the usual method of separation of variables which was applied earlier for the investigation of the wave equation $u_{tt} = a^2 \Delta u$ in Section 5-2; the situation here is completely analogous.

For this purpose we consider an auxiliary problem:

Determine a nontrivial solution of the equation

$$u_t - a^2 \Delta u = 0 ,$$

which satisfies the homogeneous boundary condition

$$u|_\Sigma = 0 \tag{6-2.2}$$

and is representable in the form

$$u(M, t) = v(M)T(t) .$$

By separation of variables in the above manner we arrive at the following equations for $v(M)$ and $T(t)$:

$$\begin{aligned} \Delta v + \lambda v &= 0 \quad \text{in} \quad T , \\ v &= 0 \quad \text{on} \quad \Sigma , \end{aligned} \tag{6-2.3}$$

and

$$T' + a^2 \lambda T = 0 . \tag{6-2.4}$$

For the function v, we have an eigenvalue problem which was already seen in the study of the vibrations of a bounded spatial region (see Section 5-3.1).

Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be the eigenvalues and $v_1, v_2, \ldots, v_n$ be the corresponding eigenfunctions of problem (6-2.3). The functions v_n form an orthogonal system.

The corresponding functions $T_n(t)$ have the form

$$T_n(t) = C_n e^{-a^2 \lambda_n t} ,$$

so that the auxiliary problem has the nontrivial solution

$$u_n(M, t) = C_n v_n(M) e^{-a^2 \lambda_n t} . \tag{6-2.5}$$

The general solution of the original problem can, therefore, be written in the form

$$u(M, t) = \sum_{n=1}^{\infty} C_n e^{-a^2 \lambda_n t} v_n(M) . \tag{6-2.6}$$

From the initial condition

$$u(M, 0) = \varphi(M) = \sum_{n=1}^{\infty} C_n v_n(M) , \tag{6-2.7}$$

we obtain the coefficients

$$C_n = \frac{\int_T \varphi(M') v_n(M') \, d\tau_{M'}}{N_n} ,$$

where

$$N_n = \int_T v_n^2(M')\, d\tau_{M'}$$

denotes the norm of the function v_n. The function (6-2.6) is, therefore, actually a solution of the problem.

The solution of the inhomogeneous equation

$$u_t - a^2 \Delta u = f(M, t)\,, \qquad \left(f = \frac{F}{c\rho}\right) \tag{6-2.8}$$

with homogeneous boundary and initial conditions can likewise be obtained with the help of this method.

To this end we set as usual

$$u(M, t) = \sum_{n=1}^{\infty} T_n(t) v_n(M) \tag{6-2.9}$$

and expand the function $f(M, t)$ with respect to the eigenfunctions $v_n(M)$. The result is

$$f(M, t) = \sum_{n=1} f_n(t) v_n(M)\,, \qquad f_n(t) = \frac{1}{N_n} \int_T f(M', t) v_n(M')\, d\tau_{M'}\,. \tag{6-2.10}$$

For the determination of $T_n(t)$ we then find the equation

$$T_n' + a^2 \lambda_n T_n = f_n(t) \tag{6-2.11}$$

with the initial condition $T_n(0) = 0$, provided $u(M, 0) = 0$. This equation is solved by the function

$$T_n(t) = \int_0^t e^{-a^2 \lambda_n (t-\tau)} f_n(\tau)\, d\tau\,. \tag{6-2.12}$$

Thus we obtain

$$u(M, t) = \int_0^t \int_T \left\{ \sum_{n=1}^{\infty} e^{-a^2 \lambda_n (t-\tau)} \frac{v_n(M) v_n(M')}{N_n} \right\} f(M', \tau)\, d\tau_{M'}\, d\tau\,. \tag{6-2.13}$$

The expression in brackets corresponds to Green's function of an instantaneous source of intensity $Q = c\rho$, which at time τ is at the point M':

$$G(M, t, M', \tau) = \sum_{n=1}^{\infty} \frac{v_n(M) v_n(M')}{N_n} e^{-a^2 \lambda_n (t-\tau)}\,. \tag{6-2.14}$$

The solution $\bar{u}$ of the first boundary-value problem for the heat-conduction equation with an inhomogeneous boundary condition $\bar{u}|_\Sigma = \mu$ can easily be converted to the solution u of the inhomogeneous equation with the homogeneous boundary condition $u|_\Sigma = 0$, if we set

$$\bar{u} = u + \Phi\,, \tag{6-2.15}$$

where Φ is an arbitrary (sufficiently smooth) function which assumes the value μ on Σ (see Section 3-2). The most frequently occurring case of a constant boundary value $\mu_0 = \text{const}$ can be transformed with the help of the function

$$\bar{u} = u + \mu_0 , \qquad (\psi = \text{const} = \mu_0)$$

into the corresponding problem with a homogeneous boundary condition; therefore $\bar{u}$ is the deviation from the stationary solution.

From these considerations, it follows that the fundamental difficulty in the investigation of heat propagation in bounded regions consists of the determination of the eigenvalues and eigenfunctions for the given region.

The form (6-2.6) of a solution obtained by separation of variables proves to be particularly suited for the investigation of sufficiently wide progressive stages of the propagation processes for large t. The eigenvalue λ_n increases rapidly with the number n for an arbitrary region. Consequently, for $t > 0$ the series converges rapidly, so that after a certain time the first term different from 0 dominates in comparison with all the remaining terms:

$$u(M, t) \approx C_1 v_1(M) e^{-a^2 \lambda_1 t} . \qquad (6\text{-}2.16)$$

This result corresponds to the physical fact that independent of the initial distribution, after a fixed time, a "regular" temperature process develops in the body for which the "profile" of the temperature changes only slightly with time and the amplitude decreases exponentially with increasing time. The nonstationary methods for the determination of temperature conductivity coefficients depend on this process. By measuring the temperature of the body at an arbitrary point M_0, we find

$$\ln |u(M_0, t)| \approx - a^2 \lambda_1 t + \ln |C_1 v_1(M)| . \qquad (6\text{-}2.17)$$

The graphical form of this function is represented by a straight line with slope equal to $-a^2\lambda_1$ at a fixed time. If we know the magnitude of λ_1, which naturally depends on the shape of the region, we can then determine the temperature conductivity.

2. Cooling of a circular cylinder

We shall now investigate the problem of cooling an infinitely long cylinder of radius r_0 which possesses a fixed initial temperature, while the temperature on its surface is equal to 0. Furthermore, we shall assume that the initial temperature does not depend on z (the z-axis is in the direction of the axis of the cylinder). The subsequent temperature states are then similarly independent of z. The temperature state can change, therefore, only in the cross section S of the cylinder. In this cross section we choose a polar coordinate system whose pole coincides with the center of the circle S. Our problem then leads to the determination of the function $u(r, \varphi, t)$ which satisfies the equation

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \varphi^2} = \frac{1}{a^2}\frac{\partial u}{\partial t} \qquad (6\text{-}2.18)$$

with initial condition

$$u(r, \varphi, 0) = \psi(r, \varphi) \qquad (6\text{-}2.19)$$

and the boundary condition

$$u(r_0, \varphi, t) = 0 . \tag{6-2.20}$$

As stated earlier, the solution of such a problem can be represented in the form

$$u = \Sigma C e^{-a^2\lambda t} v(M) . \tag{6-2.21}$$

Here the summation extends over all the eigenfunctions of the problems

$$\frac{\partial^2 v}{\partial r^2} + \frac{1}{r}\frac{\partial v}{\partial r} + \frac{1}{r^2}\frac{\partial^2 v}{\partial \varphi^2} + \lambda v = 0 , \qquad v(r_0, \varphi) = 0 . \tag{6-2.22}$$

We have already investigated this eigenvalue problem in connection with the vibrations of a circular membrane (see Section 5-3). We found that each eigenvalue

$$\lambda_{mn} = \left(\frac{\mu_m^{(n)}}{r_0}\right)^2 \tag{6-2.23}$$

corresponds to two eigenfunctions

$$\bar{v}_{nm} = J_n\left(\frac{\mu_m^{(n)}}{r_0} r\right) \cos n\varphi ,$$

and

$$\bar{\bar{v}}_{nm} = J_n\left(\frac{\mu_m^{(n)}}{r_0} r\right) \sin n\varphi , \tag{6-2.24}$$

whose norms are equal to

$$N_{nm} = \frac{\varepsilon_n r_0^2}{2}[J_n'(\mu_m^{(n)})]^2 , \qquad \varepsilon_n = \begin{cases} 1 & \text{for } n \neq 0 , \\ 2 & \text{for } n = 0 . \end{cases} \tag{6-2.25}$$

Here $\mu_m^{(n)}$ denotes the m th root of the equation

$$J_n(\mu) = 0 . \tag{6-2.26}$$

By using the expression for v and λ, we obtain

$$u(r, \varphi, t) = \sum_{n=0}^{\infty} \sum_{m=1}^{\infty} (\bar{C}_{nm} \cos n\varphi + \bar{\bar{C}}_{nm} \sin n\varphi) J_n\left(\frac{\mu_m^{(n)}}{r_0} r\right) e^{-a^2(\mu_m^{(n)}/r_0)^2 t} , \tag{6-2.27}$$

where the coefficients $\bar{C}_{nm}$ and $\bar{\bar{C}}_{nm}$ are calculated from the initial values according to

$$\bar{C}_{nm} = \frac{\int_0^{r_0}\int_0^{2\pi} \Phi(r, \varphi) J_n\left(\frac{\mu_m^{(n)}}{r_0} r\right) \cos n\varphi r \, d\varphi \, dr}{\frac{\pi r_0^2}{2} \varepsilon_n [J_n'(\mu_m^{(n)})]^2} ,$$

$$\bar{\bar{C}}_{nm} = \frac{\int_0^{r_0}\int_0^{2\pi} \Phi(r, \varphi) J_n\left(\frac{\mu_m^{(n)}}{r_0} r\right) \sin n\varphi r \, d\varphi \, dr}{\frac{\pi r_0^2}{2} [J_n'(\mu_m^{(n)})]^2} , \tag{6-2.28}$$

$$\varepsilon_n = \begin{cases} 1 & \text{for } n \neq 0 , \\ 2 & \text{for } n = 0 . \end{cases}$$

If the initial temperature φ depends only on r, then the double series (6-2.27) reduces to the single series

$$u(r, t) = \sum_{m=1}^{\infty} C_m J_0\left(\frac{\mu_m^{(0)}}{r_0} r\right) e^{-a^2 \mu_m / r_0^2 t}, \tag{6-2.29}$$

where

$$C_m = \frac{2\int_0^{r_0} \varphi(r) J_0\left(\frac{\mu_m^{(0)}}{r_0} r\right) r\, dr}{r_0^2 [J_1(\mu_m^{(0)})]^2}, \qquad (J_1 = -J_0'), \tag{6-2.30}$$

while $\mu_m^{(0)}$ denotes the mth root of the equation $J_0(\mu) = 0$.

We now wish to discuss, in somewhat more detail, the cooling of a uniformly heated cylinder. Let the temperature on the surface of the cylinder once more equal 0. If

$$u(r, 0) = \varphi = u_0$$

is the initial temperature, then

$$C_m = \frac{2u_0 \int_0^{r_0} J_0\left(\frac{\mu_m^{(0)}}{r_0} r\right) r\, dr}{r_0^2 [J_1(\mu_m^{(0)})]^2} = \frac{2u_0 \frac{r_0}{\mu_m^{(0)}} \left[r J_1\left(\frac{\mu_m^{(0)}}{r_0} r\right)\right]_0^{r_0}}{r_0^2 [J_1(\mu_m^{(0)})]^2} = \frac{2u_0}{\mu_m^{(0)} J_1(\mu_m^{(0)})}, \tag{6-2.30'}$$

since $\alpha J_0(\alpha) = [\alpha J_1(\alpha)]'$. Thus we obtain

$$\frac{u(r, t)}{u_0} = \sum_{m=1}^{\infty} \frac{2J_0(\mu_m^{(0)} \rho)}{\mu_m^{(0)} J_1(\mu_m^{(0)})} e^{-(\mu_m^{(0)})^2 \theta}, \qquad \left(\rho = \frac{r}{r_0}, \quad \theta = \frac{a^2 t}{r_0^2}\right). \tag{6-2.31}$$

In the table of cylindrical functions (see Vol. I, Appendix), the numerical values for the roots $\mu_m^{(0)}$ as well as for $J_1(\mu_m^{(0)})$ are given. In particular

$$\mu_1^{(0)} = 2.40, \qquad J_1(\mu_1^{(0)}) = 0.52,$$
$$\mu_2^{(0)} = 5.52, \qquad J_1(\mu_2^{(0)}) = -0.34.$$

The series (6-2.31) converges very rapidly, so that we can limit ourselves to the first term for large t. In particular, on the axis of the cylinder we have

$$\frac{u(r, t)}{u_0} = \frac{2}{2.40 \times 0.52} e^{-(2.40)^2 \theta} = 1.60\, e^{-5.76\theta}, \qquad \left(\theta = \frac{a^2 t}{r_0}\right). \tag{6-2.32}$$

3. Determination of the critical dimensions

In Chapter 3 we showed that the process of diffusion of an unstable gas, whose velocity of decomposition is proportional to the concentration, leads to the equation

$$u_t = a^2 \Delta u + \beta u, \qquad (\beta < 0). \tag{6-2.33}$$

Diffusion processes are of great interest when chain reactions are present. Chain reactions are characterized by the fact that the number of particles of the diffusing material increase by reaction with the surrounding medium.

Thus, for example, a nuclear fission results from the collision of neutrons with the active uranium nucleus, which leads to the occurrence of new neutrons whose number is greater than one. These neutrons, in turn, react with active neuclei and lead, in the same way, to fission whereby new neutrons again occur. In this way the number of neutrons increases. Here we are dealing with a typical chain reaction.

If this process is treated approximately as a diffusion process, we arrive at the equation

$$u_t = a^2 \Delta u + \beta u , \qquad (\beta > 0) , \tag{6-2.33'}$$

since the chain reaction is equivalent to the existence of sources of diffusing materials (neutrons) proportional to the concentration (neutron density).

We consider the following problem:

Determine the solution of the equation

$$u_t = a^2 \Delta u + \beta u \quad \text{in the interior of } T, \tag{6-2.33''}$$

which satisfies the initial condition

$$u(M, 0) = \varphi(M) \tag{6-2.34}$$

and the boundary condition

$$u|_\Sigma = 0 . \tag{6-2.35}$$

With the help of the substitution

$$u(M, t) = \bar{u}(M, t)e^{\beta t} , \tag{6-2.36}$$

Eq. (6-2.33'') leads to Eq. (6-2.1). The initial and boundary conditions thus remain unchanged. Therefore the desired function $u(M, t)$ has the form

$$u(M, t) = \sum_{n=1}^{\infty} C_n e^{(\beta - a^2\lambda_n)t} v_n(M) . \tag{6-2.37}$$

The coefficients C_n are fixed by the initial function according to formula (6-2.10). In case $\beta < 0$ (diffusion with decomposition), the exponents of the series (6-2.37) are larger than the corresponding exponents in the series (6-2.6). This means that for decomposition phenomena the concentration decreases relatively faster than in the case of pure diffusion ($\beta = 0$). In case $\beta > 0$ (diffusion with reproduction) if, at least for one exponent, $\beta - a^2\lambda > 0$, i.e., $\beta > a^2\lambda_1$ holds, then in general ($C_1 \neq 0$) there is an exponential increase of the concentration with time (chain reaction). The quantity β is characteristic of the substance in question (reproduction coefficient), while λ is essentially dependent on the form and dimensions of the region. We shall say that a region $T_{k\tau}$ possesses critical dimensions for a given β, when $\lambda_1 = \beta/a^2$. In the following we shall calculate the critical dimension of a slab, an infinite cylinder and a sphere.

1. *The slab* $0 \leqq x \leqq l$. If we consider the problem as a one-dimensional problem, then

$$\lambda_n = \left(\frac{\pi n}{l}\right)^2 \quad \text{and} \quad \lambda_1 = \frac{\pi^2}{l^2} .$$

(see Section 2-2). The critical thickness l_{kr}, resulting from the concentration u by an avalanching increase, can also be determined from the expression

$$l_{kr} = \frac{a\pi}{\sqrt{\beta}} \approx \frac{3.14a}{\sqrt{\beta}}, \qquad (\beta > 0). \tag{6-2.38}$$

2. *The infinite cylinder.* We consider this problem as a two-dimensional problem. Hence we see that the smallest eigenvalue λ corresponds to a radially symmetric eigenfunction and that this eigenvalue is equal to

$$\lambda_1^{(0)} = \left(\frac{\mu_1^{(0)}}{r_0}\right)^2 a, \qquad (\mu_2^{(0)} = 2.4048).$$

Thus we obtain for the critical diameter the formula

$$d_{kr} = \frac{2\mu_1^{(0)} a}{\sqrt{\beta}} \approx \frac{4.80a}{\sqrt{\beta}}. \tag{6-2.39}$$

3. *The sphere.* The smallest eigenvalue here corresponds to a spherically symmetric eigenfunction. It is equal to

$$\lambda_1 = \left(\frac{\pi}{R}\right)^2,$$

so that for the critical diameter D_{kr}

$$D_{kr} = \frac{2\pi a}{\sqrt{\beta}} \approx \frac{6.28a}{\sqrt{\beta}} \tag{6-2.40}$$

holds.

6-3. BOUNDARY-VALUE PROBLEMS FOR REGIONS WITH VARIABLE BOUNDARIES

1. Green's formula for the heat-conduction equation and Green's function

For the heat-conduction equation boundary-value problems arise for regions whose boundaries depend on the time t.

For the purpose of simplicity we shall consider this problem only for the equation with one position variable

$$L(u) = a^2 u_{xx} - u_t = 0, \tag{6-3.1}$$

although the following investigation can also carry over to the case of many variables.

First we start with a region $BAEF$ which is bounded by the two characteristics AB and EF ($t = \text{const}$) as well as by the curves

$$x = \chi_1(t) \quad \text{for } AE$$

and

$$x = \chi_2(t) \quad \text{for } BF. \quad \text{(Fig. 83)}$$

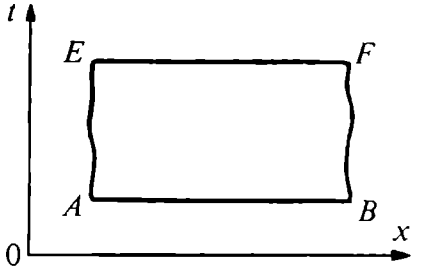


FIG. 83

The first boundary-value problem for this region consists of the determination of that solution of the heat-conduction equation (6-3.1) which satisfies the initial and boundary conditions

$$u = \varphi(x) \text{ on } AB\,,$$
$$u|_{x=\chi_1(t)} = \mu_1(t)\,, \qquad u|_{x=\chi_2(t)} = \mu_2(t)\,. \tag{6-3.2}$$

It follows immediately from the principle of the maximum that this problem can have, at most, one solution. We formulate the other boundary-value problems correspondingly.

We shall now establish Green's formula for Eq. (6-3.1) and, therefore, give an integral representation of the solution of this problem.

For this purpose we consider the differential expression

$$\boldsymbol{M}(v) = a^2 \frac{\partial^2 v}{\partial x^2} + \frac{\partial v}{\partial t}\,. \tag{6-3.3}$$

If we integrate

$$\psi \boldsymbol{L}(\varphi) - \varphi \boldsymbol{M}(\psi)$$

over a region $PABQ$ (Figure 84) [here $\varphi(x, t)$ and $\psi(x, t)$ are arbitrarily often

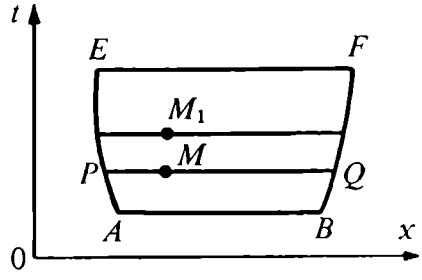


FIG. 84

differentiable functions], then we obtain after a simple transformation

$$\iint [\psi \boldsymbol{L}(\varphi) - \varphi \boldsymbol{M}(\psi)]\, dx\, dt = \int \varphi\psi\, dx + a^2 \left(\psi \frac{\partial \varphi}{\partial x} - \varphi \frac{\partial \psi}{\partial x}\right) dt\,,$$

where the integral on the right side is extended over the closed contour $PABQ$. If $\boldsymbol{L}(\varphi) = 0$ and $\boldsymbol{M}(\psi) = 0$, then after splitting up the integral on the right we obtain

$$\int_{PQ} \varphi\psi\,dx \quad \int_{AB} \varphi\psi\,dx + \int_{BQ}\left[\varphi\psi\,dx + a^2\left(\psi\frac{\partial\varphi}{\partial x} \quad \varphi\frac{\partial\psi}{\partial x}\right)dt\right] \int_{AP}\left[\varphi\psi\,dx + a^2\left(\psi\frac{\partial\varphi}{\partial x} \quad \varphi\frac{\partial\psi}{\partial x}\right)dt\right]. \tag{6-3.4}$$

Now let $\varphi(x, t) \quad u(x, t)$ be any solution of the heat-conduction equation $L(u) - 0$, while $\psi \quad G_0(x, t, \xi, \tau)$ is the Green's function of this equation on the unbounded straight lines:

$$G_0(x, t, \xi, \tau) \quad \frac{1}{2\sqrt{\pi a^2(t \quad \tau)}} e^{(x \quad \xi)^2/4a^2(t \quad \tau)}. \tag{6-3.5}$$

$G_0(x, t, \xi, \tau)$ is usually called the fundamental solution of the heat-conduction equation. It satisfies the equation $L(G_0) \quad 0$ with respect to the variables x, t and the adjoint equation $M(G_0) \quad 0$ with respect to the variables ξ, τ.

Furthermore, let $M(x, t)$ be any fixed point in the interior of the region $EAFH$ in which we want to determine the value of the function $u(x, z)$, whereas M_1 is a point with coordinates $x, t + h$, $h > 0$. If we now place the characteristic PQ through the point M, replace x by ξ and t by τ in formula (6-3.4), and apply the resulting formula and the functions

$$\varphi \quad u(x, t) \quad \text{and} \quad \psi \quad G_0(x, t + h, \xi, \tau) \tag{6-3.6}$$

to the region $AB QP$ (Figure 84), we obtain

$$\int_{PQ} \frac{e^{(x \quad \xi)^2/4a^2h}}{2\sqrt{\pi a^2 h}} u(\xi, t)\,d\xi \quad \int_{PABQ} u(\xi, \tau)G_0(x, t + h, \xi, \tau)\,d\xi + a^2\left(G_0\frac{\partial u}{\partial\xi} \quad u\frac{\partial G_0}{\partial\xi}\right)d\tau. \tag{6-3.7}$$

By passing to the limit as $h \to 0$ and by bearing in mind the continuity of the functions $G_0(x, t + h, \xi, \tau)$ and $\partial G_0/\partial\xi$ on $PABQ$ with respect to h as well as the equation[14]

$$\lim_{h \to 0} \int_{PQ} \frac{e^{(x \quad \xi)^2/4a^2h}}{2\sqrt{\pi a^2 h}} u(\xi, t)\,d\xi \quad u(x, t), \tag{6-3.8}$$

where (x, t) lies on PQ, we obtain the desired integral formula

$$u(x, t) \quad \int_{PABQ} u(\xi, \tau)G_0(x, t, \xi, \tau)\,d\xi + \int_{BQ+PA} a^2\left(G_0\frac{\partial u}{\partial\xi} \quad u\frac{\partial G_0}{\partial\xi}\right)d\tau. \tag{6-3.9}$$

Each solution of the heat-conduction equation can be represented by formula (6-3.9). We write the formula in greater detail:

$$u(x, t) \quad \int_{PABQ} \frac{e^{(x \quad \xi)^2/4a^2(t \quad \tau)}}{2\sqrt{\pi a^2(t \quad \tau)}} u(\xi, \tau)\,d\xi + a^2\int_{BQ+PA} \frac{e^{(x \quad \xi)^2/4a^2(t \quad \tau)}}{2\sqrt{\pi a^2(t \quad \tau)}}\frac{\partial u}{\partial\xi}\,d\tau \quad a^2\int_{BQ+PA} u(\xi, \tau)\frac{\partial}{\partial\xi}\left(\frac{e^{(x \quad \xi)^2/4a^2(t \quad \tau)}}{2\sqrt{\pi a^2(t \quad \tau)}}\right)d\tau. \tag{6-3.9'}$$

Obviously, this formula is not yet useful in the solution of the boundary-

[14] See Section 3-3.

value problem, since for the calculation of the right side, not only u, but also $\partial u/\partial \xi$ must be known along the curves AE and BF.

Using transformations similar to those which we carried out for Laplace's equation by introducing Green's function, we can eliminate the derivative $\partial u/\partial \xi$ from this formula.

Let v be an arbitrary solution of the adjoint equation $M(v) = 0$ which is equal to zero on PQ, and let u be a solution of the heat-conduction equation $L(u) = 0$. By applying formula (6-3.4) to the functions v and u in the region $PABQ$, we obtain

$$0 = \int_{PABQ} \left[u(\xi, \tau)v(\xi, \tau)\, d\xi + a^2 \left(v\frac{\partial u}{\partial \xi} - u\frac{\partial v}{\partial \xi} \right) d\tau \right]. \tag{6-3.10}$$

If we now subtract Eq. (6-3.10) from (6-3.9), then

$$u(x, t) = \int_{PABQ} \left[u(\xi, \tau)G(x, t, \xi, \tau)\, d\xi + a^2 \left(G\frac{\partial u}{\partial \xi} - u\frac{\partial G}{\partial \xi} \right) d\tau \right], \tag{6-3.11}$$

where

$$G(x, t, \xi, \tau) = G_0(x, t, \xi, \tau) - v\ . \tag{6-3.12}$$

If we choose v such that $G = 0$ on PA and BQ, we obtain the integral representation for $u(x, t)$ in the form

$$u(x, t) = \int_{AB} u(\xi, \tau)G(x, t, \xi, \tau)\, d\xi + a^2 \int_{AP} u\frac{\partial G}{\partial \xi}\, d\tau - a^2 \int_{BQ} u\frac{\partial G}{\partial \xi}\, d\tau\ . \tag{6-3.13}$$

Formula (6-3.13) gives the solution for boundary-value problems on whose boundaries the values of the function u are prescribed on AP and BQ.

We now study Green's function as defined by formula (6-3.10), where the function $v(\xi, \tau)$ was characterized by the following properties:

1. $v(\xi, \tau)$ is defined in the region $PABQ$ and for $\tau < t$ satisfies the adjoint equation $M(v) = 0$.
2. $v = 0$ on PQ, i.e., for $\tau = t$.
3. $v(\xi, \tau) = -G_0(x, t, \xi, \tau)$ on PA and QB.

On the basis of these conditions, v is dependent on the parameters x, t, thus $v = v(x, t, \xi, \tau)$. Consequently, for the determination of v we must solve a boundary-value problem for the equation $M(v) = 0$. This problem is equivalent to the solution of a boundary-value problem of type (6-3.2) for the equation $L(u) = 0$, as we can easily see by changing the sign of τ. Therefore, the principal difficulty with the representation of the function $u(x, t)$ by means of formula (6-3.11) as a solution of the boundary-value problem (6-3.2) lies in the determination of the function $v(x, t, \xi, \tau)$.

We consider the function $\bar{v}(x, t, \xi, \tau)$ defined by the following conditions: (a) $\bar{v}(x, t, \xi, \tau)$ is defined in the region $PABQ$ for $t > \tau$ and satisfies the heat-conduction equation $L(v) = 0$ as a function of the variables x, t; (b) $\bar{v} = 0$ on AB, i.e., for $t = \tau$; (c) $\bar{v} = -G_0$ on AP and BQ.

In the following demonstration we shall prove $v(x, t, \xi, \tau) = \bar{v}(x, t, \xi, \tau)$.

For this purpose we consider the function $\bar{G}(x, t, \xi, \tau) = G_0 + \bar{v}$. Obviously, for an arbitrary solution $\bar{u}$ of equation $M = 0$, the formula analogous to (6-3.9)

$$\bar{u}(\xi, \tau) = \int_{BQPA} \bar{u}G_0\, dx + a^2\left(\bar{u}\frac{\partial G_0}{\partial x} - G_0\frac{\partial u}{\partial x}\right)dt \qquad (6\text{-}3.9'')$$

is valid, as is the formula analogous to (6-3.13)

$$\bar{u}(\xi, \tau) = \int_{PQ} \bar{u}\bar{G}\, dx + a^2\int_{BQ} u\frac{\partial \bar{G}}{\partial x}dt - a^2\int_{AP} \bar{u}\frac{\partial \bar{G}_0}{\partial x}\, dt\,. \qquad (6\text{-}3.13')$$

These formulas can be obtained from (6-3.9) and (6-3.13) by changing the signs of τ, since by this change, equation $\boldsymbol{M} = 0$ is transformed into the equation $\boldsymbol{L} = 0$.

By applying formula (6-3.13) to the region $PRSQ$ (Figure 85) where RS is

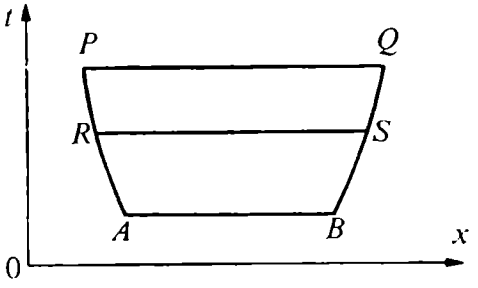


FIG. 85

the straight line segment corresponding to the ordinate θ, $t > \theta > \tau$ and to the continuous solution $u(x, t) = \bar{G}(x, t, \xi, \tau)$ of equation $\boldsymbol{L}(u) = 0$, we obtain

$$\bar{G}(x, t, \xi, \tau) = \int_{RS} \bar{G}(x', \theta, \xi, \tau)G(x, t, x', \theta)\, dx'\,,$$

since the integrals along RP and SQ on the basis of the requirement C are equal to zero.

Further, if we apply formula (6-3.13′) to the region $ARSB$ and to the continuous solution $u(\xi, \eta) = G(x, t, \xi, \eta)$ of equation $\boldsymbol{M} = 0$ in this region, we obtain

$$G(x, t, \xi, \tau) = \int_{RS} G(x, t, x', \theta)\bar{G}(x', \theta, \xi, \tau)\, dx'\,,$$

since the integrals over BS and AR are equal to zero. By comparing the last two formulas we obtain

$$G(x, t, \xi, \tau) \equiv \bar{G}(x, t, \xi, \tau)\,.$$

This equation shows that G as a function of x, t possesses a singularity characteristic of the Green's function for $t = \tau$ and $x = \xi$.

Furthermore, we recognize from this formula that G for $t = \tau$ and $x \neq \xi$ is equal to zero, satisfies the equation $\boldsymbol{L} = 0$ in the interior of $APQB$ and vanishes on AP and BQ. Because of these properties, G is called Green's function for the heat-conduction equation for region $APQB$.

From our reasoning we arrive at the following conclusion. Every solution of the heat-conduction equation can be represented by Eq. (6-3.13) by means of the corresponding Green's function.

If the inhomogeneous equation $Lu = f(x, t)$ is given, then in formula (6 3.13) we must add to the right side the following term

$$\int_S \int G(x, t, \xi, \tau) f(\xi, \tau)\, d\xi\, d\tau .$$

2. Solution of the boundary-value problem

Formula (6 3.13) obtained above gives the solution of the boundary-value problem for a bounded interval with time dependent end points. If the end points of the segment AB are independent of t, then the curves AE and BF reduce to two straight-line elements parallel to the t-axis. The region S in this case, therefore, has the form of a rectangle whose sides are parallel to the axis of the coordinates. From the general formula (6 3.11) and by passage to the limit, we can now obtain the Poisson formula for the solution of the heat-conduction equation with prescribed initial conditions on the infinite straight line.

We assume that in the part of the region which is bounded by the two characteristics $t = 0$ and $t = \delta$ (these pass through the points A and E) the inequalities

$$u(x, t)e^{-Kx^2} < N \quad \text{and} \quad \frac{\partial u}{\partial x} e^{-Kx^2} < N \qquad (A)$$

are satisfied for the functions u and u_x; thus let $K > 0$ and $N > 0$ be fixed constants. We then replace the segment BQ (Figure 86) by the straight-line

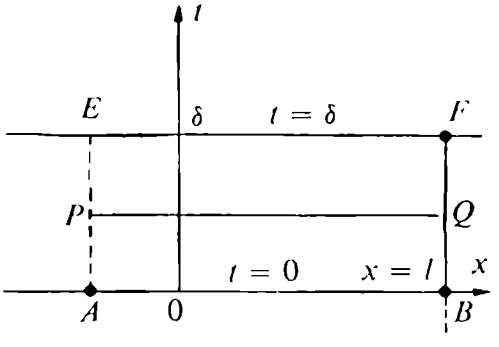


FIG. 86

element $x = l'$. Let l be a positive number which we shall later increase without limit. Thus we arrive at formula (6-3.9) which we write in the form

$$u(x, t) = \int_{PABQ} \frac{e^{-(x-\xi)^2/4a^2(t-\tau)}}{2\sqrt{\pi a^2(t-\tau)}} \left[u(\xi, \tau)\, d\xi + a^2 \frac{\partial u}{\partial \xi}\, d\tau - u(\xi, \tau) \frac{x - \xi}{2(t-\tau)} d\tau \right].$$

We shall now show that, as $l \to \infty$, the expression

$$\int_{BQ} \frac{e^{-(x-l)^2/4a^2(t-\tau)}}{2\sqrt{\pi a^2(t-\tau)}} \left[a^2 \left(\frac{\partial u}{\partial \xi} \right)_{\xi = l} - u(l, \tau) \frac{x - l}{2(t-\tau)} \right] d\tau$$

$$\int_0^t a^2 \left(\frac{\partial u}{\partial \xi} \right)_{\xi = l} \frac{e^{-(x-l)^2/4a^2(t-\tau)}}{2\sqrt{\pi a^2(t-\tau)}}\, d\tau - \int_0^t u(l, \tau) \frac{e^{-(x-l)^2/4a^2(t-\tau)}(x - l)\, d\tau}{2\sqrt{\pi a^2(t-\tau)}(t-\tau)} = I_1 + I_2$$

approaches zero.

For large values of l the estimate

$$|I_1| \leqq \frac{Na^2}{2\sqrt{\pi a^2}} e^{Kl^2 - l^2/16a^2\delta} \int_0^t \frac{dt}{\sqrt{t-\tau}}$$

holds for $x < l/2$ and $(t-\tau) < \delta$. Obviously, as $l \to \infty$, $|I_1|$ approaches zero, since N is a fixed number while δ can be chosen arbitrarily small. In particular, we find that

$$K < \frac{1}{16a^2\delta} .$$

Correspondingly, we prove $|I_2| \to 0$ as $l \to \infty$. If for the function $u(x, t)$ and its partial derivative $\partial u/\partial x$, the inequality (A) is also satisfied for negative x, then for AE we can choose the straight segment $x = -l$. By repeating the above reasoning, we find that the integral over PA in formula (6-3.9) approaches zero by passage to the limit. Hence, we arrive at the Poisson formula[15] (Section 3-3)

$$u(x, t) = \frac{1}{2\sqrt{\pi a^2}} \int_{-\infty}^{+\infty} \frac{e^{-(x-\xi)^2/4a^2t}}{\sqrt{t}} u(\xi, 0)\, d\xi .$$

If we consider a one-sided unbounded region and assume that for Green's function $G(x, t, \xi, \tau)$ the inequality (A) is satisfied, then by analogous reasoning we obtain

$$u(x, t) = -\int_{PA} a^2 \mu(\tau) \frac{\partial G}{\partial \xi}\bigg|_{\xi = x_A} d\tau + \int_{x_A}^{\infty} \varphi(\xi) G(x, t, \xi, 0)\, d\xi , \qquad (6\text{-}3.14)$$

where

$$\mu(t) = u(x_A, t) \quad \text{and} \quad \varphi(x) = u(x, 0) .$$

Green's function for the one-sided unbounded straight line $x \geqq 0$ can be constructed by means of the reflection method. It is equal to

$$G(x, t, \xi, \tau) = \frac{1}{2\sqrt{\pi a^2 (t-\tau)}} [e^{-(x-\xi)^2/4a^2(t-\tau)} - e^{-(x+\xi)^2/4a^2(t-\tau)}] ,$$

since it is representable in the form (6-3.12), satisfies the heat-conduction equation with respect to the variables x, t, and is equal to 0 for $x = 0$:

$$G(0, t, \xi, \tau) = 0 .$$

Now we calculate

$$\frac{\partial G}{\partial \xi}_{\xi=0} = \frac{x}{2\sqrt{\pi}\,[a^2(t-\tau)]^{3/2}} e^{-x^2/4a^2(t-\tau)} .$$

By substitution of these values into Eq. (6-3.14) we obtain the formula

$$u(x, t) = \frac{1}{2\sqrt{\pi}} \int_0^{\infty} \frac{1}{\sqrt{a^2 t}} [e^{-(x-\xi)^2/4a^2t} - e^{-(x-\xi)^2/4a^2t}] \varphi(\xi)\, d\xi$$

$$+ \frac{1}{2\sqrt{\pi}} \int_0^t \frac{a^2 x}{[a^2(t-\tau)]^{3/2}} e^{-x^2/4a^2(t-\tau)} \mu(\tau)\, d\tau , \qquad (6\text{-}3.15)$$

[15] This cannot be considered as a derivation of the Poisson formula since the derivation of formula (6-3.9) is based on the Poisson formula.

which defines the function $u(x, t)$ satisfying the equation

$$u_t = a^2 u_{xx} , \qquad (0 < x < \infty , \quad t > 0)$$

and the conditions

$$\begin{aligned} u(x, 0) &= \varphi(x) , \\ u(0, t) &= \mu(t) , \end{aligned} \qquad (0 < x < \infty) .$$

3. Green's function for an interval

The solution of the heat-conduction equation in a bounded interval $0 < x < l$ is given by formula (6-3.11). If the arcs PA and BQ are replaced by the corresponding straight-line segments and the origin is placed at the point A, then this formula assumes the form

$$u(x, t) = a^2 \int_0^t \left.\frac{\partial G}{\partial \xi}\right|_{\xi=0} \mu_1(\tau)\, d\tau - a^2 \int_0^t \left.\frac{\partial G}{\partial \xi}\right|_{\xi=l} \mu_2(\tau)\, d\tau + \int_0^l G(x, t, \xi, 0)\varphi(\xi)\, d\xi .$$

Here

$$\mu_1(t) = u(0, t) , \qquad \mu_2(t) = u(l, t) , \qquad \varphi(x) = u(x, 0) .$$

Green's function $G(x, t, \xi, \tau)$ for an interval can be obtained with the help of the reflection method. Here we assume that the positive sources are at the points $2nl + \xi$ and the negative sources at the points $2nl - \xi$. Then the desired Green's function can be represented by

$$u(x, t) = \sum_{n=-\infty}^{\infty} [G_0(x, t, 2nl + \xi, \tau) - G_0(x, t, 2nl - \xi, \tau)] , \qquad (6\text{-}3.16)$$

where

$$G_0(x, t, \xi, \tau) = \frac{1}{2\sqrt{\pi a^2 (t - \tau)}} e^{-(x-\xi)^2/4a^2(t-\tau)}$$

is Green's function of the unbounded straight line.

It is easy to confirm that the series converges and that the boundary conditions

$$G|_{x=0} = 0 \quad \text{and} \quad G|_{x=l} = 0$$

are satisfied.

In Section 3-2 another form of Green's function was given, that is, the representation

$$G(x, t, \xi, \tau) = \frac{2}{l} \sum_{n=1}^{\infty} e^{-(n^2\pi^2/l^2)a^2(t-\tau)} \sin\frac{\pi n}{l} x \sin\frac{\pi n}{l}\xi . \qquad (6\text{-}3.17)$$

We shall now prove the equivalence of the two representations.

Formula (6-3.17) can be interpreted as the Fourier expansion of $G(x, t, \xi, \tau)$ in terms of sine functions in the interval $(0, l)$:

$$G(x, t, \xi, \tau) = \sum_{n=1}^{\infty} G_n(x, t, \tau) \sin\frac{\pi n}{l}\xi . \qquad (6\text{-}3.18)$$

For the Fourier coefficients G_n of the function G defined by the series (6-3.16) we obtain

$$G_n = \frac{2}{l}\int_0^l G(x, t, \xi, \tau)\sin\frac{\pi n}{l}\xi\, d\xi$$
$$= \frac{2}{l}\sum_{n=-\infty}^{\infty}\left\{\int_0^l G_0(x, t, 2nl + \xi_1, \tau)\sin\frac{\pi n}{l}\xi_1\, d\xi_1 - \int_0^l G_0(x, t, 2nl - \xi_2, \tau)\sin\frac{\pi n}{l}\xi_2\, d\xi_2\right\}.$$

Now if we introduce new integration variables

$$\xi' = 2nl + \xi_1 \quad \text{and} \quad \xi'' = 2nl - \xi_2,$$

we obtain

$$G_n = \frac{2}{l}\sum_{n=-\infty}^{\infty}\left\{\int_{2nl}^{(2n+1)l} G_0(x, t, \xi', \tau)\sin\frac{\pi n}{l}\xi'\, d\xi' + \int_{(2n-1)l}^{2nl} G_0(x, t, \xi'', \tau)\sin\frac{\pi n}{l}\xi''\, d\xi''\right\}.$$

From this it follows that

$$G_n = \frac{2}{l}\int_{-\infty}^{+\infty} G_0(x, t, \xi, \tau)\sin\frac{\pi n}{l}\xi\, d\xi$$
$$= \frac{2}{l}\int_{-\infty}^{+\infty}\frac{1}{2\sqrt{\pi a^2(t-\tau)}}e^{-(x-\xi)^2/4a^2(t-\tau)}\sin\frac{\pi n}{l}\xi\, d\xi.$$

Then if we set

$$\lambda = \frac{\xi - x}{2\sqrt{a^2(t-\tau)}},$$

we obtain

$$d\lambda = \frac{d\xi}{2\sqrt{a^2(t-\tau)}},$$

$$\sin\frac{\pi n}{l}\xi = \sin\frac{\pi n}{l}(x + 2\sqrt{a^2(t-\tau)}\,\lambda)$$
$$= \sin\frac{\pi n}{l}x\cos\frac{2\pi n}{l}\sqrt{a^2(t-\tau)}\,\lambda + \cos\frac{\pi n}{l}x\sin\frac{2\pi n}{l}\sqrt{a^2(t-\tau)}\,\lambda,$$

$$G_n = \frac{2}{l}\sin\frac{\pi n}{l}\frac{1}{2\sqrt{\pi}}\int_{-\infty}^{+\infty}e^{-\lambda^2}\cos\frac{2\pi n}{l}\sqrt{a^2(t-\tau)}\,\lambda\, d\lambda + \frac{2}{l}\cos\frac{\pi n}{l}x\int_{-\infty}^{+\infty}e^{-\lambda^2}\sin\frac{2\pi n}{l}\sqrt{a^2(t-\tau)}\,\lambda\, d\lambda.$$

The second integral is equal to zero since the integrand is an odd function with respect to the origin.

The first integral is a special case of the integral

$$I(\alpha, \beta) = \int_{-\infty}^{+\infty}e^{-\alpha\lambda^2}\cos\beta\lambda\, d\lambda,$$

which was calculated in Section 3-3 and is equal to

$$I(\alpha, \beta) = \frac{1}{2}\sqrt{\frac{\pi}{\alpha}}e^{-\beta^2/4\alpha^2}.$$

In our case,

$$\alpha = 1\,, \qquad \beta = \frac{2\pi n}{l}\sqrt{a^2(t-\tau)}$$

holds, so that

$$I = \frac{\sqrt{\pi}}{2} e^{-(\pi^2 n^2/l^2)a^2(t-\tau)}$$

and

$$G_n = \frac{2}{l} e^{-(\pi^2 n^2/l^2)a^2(t-\tau)} \sin\frac{\pi n}{l} x\,.$$

If we introduce this expression for the Fourier coefficients G_n into formula (6-3.18), then we obtain the second representation (6-3.17) of Green's function directly. Therefore, the equivalence of both representations (6-3.16) and (6-3.17) is proved.

6-4. HEAT POTENTIALS

1. Properties of heat potentials of single and double layers

As we have seen, the solution of the heat-conduction equation can be represented in the form

$$u(x,t) = \int_{AB} G_0 u\, d\xi - \int_{AP} G_0 u\, d\xi + \int_{BQ} G_0 u\, d\xi + a^2 \int_{BQ+PA} \left(G_0 \frac{\partial u}{\partial \xi} - u \frac{\partial G_0}{\partial \xi} \right) d\tau\,.$$

We now turn to the investigation of the individual summands of this sum and show first that these individual summands satisfy the heat-conduction equation. The first summand is a Poisson integral and therefore, as already proved, satisfies the heat-conduction equation.

In order to investigate the other summands, we next prove that for interior points of the region $PABQ$, the integral

$$V = a^2 \int_{AP} G_0 \nu\, d\tau = \frac{a^2}{2\sqrt{\pi}} \int_0 \frac{1}{\sqrt{a^2(t-\tau)}} e^{-[x-\chi_1(\tau)]^2/4a^2(t-\tau)} \nu(\tau)\, d\tau\,,$$

$$(\xi = \chi_1(\tau))\,,$$

$$W = 2a^2 \int_{AP} \frac{\partial G_0}{\partial \xi} \mu\, d\tau = \frac{1}{2a\sqrt{\pi}} \int_0^t \frac{x - \chi_1(\tau)}{(t-\tau)^{3/2}} e^{-[x-\chi_1(\tau)]^2/4a^2(t-\tau)} \mu(\tau)\, d\tau$$

satisfies the heat-conduction equation.

The derivatives of these functions with respect to the parameters x and t can be calculated by differentiation under the integral signs for $x \neq \chi_1(t)$, since the integrands and their partial derivatives with respect to x and t are continuous. For $t \to \tau$ and $x \neq \chi_1(t)$ the integrands approach zero exponentially. For example,

$$G_0(x, t, \chi_1(\tau), \tau)\mu(\tau)|_{\tau=t} = \frac{1}{2\sqrt{\pi}} \frac{1}{\sqrt{a^2(t-\tau)}} e^{-[x-\chi_1(\tau)]^2/4a^2(t-\tau)} \mu(\tau)\Big|_{\tau=t} = 0\,.$$

The differentiation with respect to the parameters x, t is, therefore, related to the function G_0 which is itself a solution of the heat-conduction equation. The remaining summands are treated analogously.

Now we consider the behavior of the functions V and W on the curve AP ($x = \chi_1(t)$). Obviously, the integral V is continuous for a passage of the points (x, t) through the curve AP, since this integral converges uniformly (see Section 4-5). On the other hand, as we shall prove at once, W undergoes a jump by passage through the curve AP, that is,

$$W|_{x=\chi_1(t)+0} = W|_{x=\chi_1(t)} + \mu(t) ,$$
$$W|_{x=\chi_1(t)-0} = W|_{x=\chi_1(t)} - \mu(t) .$$

We carry out the proof for this by assuming the differentiability of $\chi_1(t)$ and $\mu(t)$.

First, we consider W for constant density $\mu(t) = \mu_0$:

$$W^0(x, t) = \frac{1}{2a\sqrt{\pi}} \int_{t_0}^{t} \frac{x - \chi_1(\tau)}{(t-\tau)^{3/2}} e^{-[x-\chi_1(\tau)]^2/4a^2(t-\tau)} \mu_0 \, d\tau$$

and besides, the integral

$$\tilde{V}^0(x, t) = \frac{1}{2a\sqrt{\pi}} \int_{t_0}^{t} \frac{2\chi_1'(\tau)}{\sqrt{t-\tau}} e^{-[x-\chi_1(\tau)]^2/4a^2(t-\tau)} \mu_0 \, d\tau$$

is continuous, on the basis of the above statement, at the points of the arc AP.

For the difference $W^0 - \tilde{V}^0$ we then obtain

$$W^0(x, t) - \tilde{V}^0(x, t) = \frac{1}{2a\sqrt{\pi}} \int_{t_0}^{t} e^{-[x-\chi_1(\tau)]^2/4a^2(t-\tau)} \left[\frac{x - \chi_1(\tau)}{(t-\tau)^{3/2}} - \frac{2\chi_1'(\tau)}{\sqrt{t-\tau}} \right] \mu_0 \, d\tau$$
$$= \mu_0 \frac{2}{\sqrt{\pi}} \int_{[x-\chi_1(t_0)]/2\sqrt{a^2(t-t_0)}}^{\alpha_0} e^{-\alpha^2} \, d\alpha , \qquad \left(\alpha = \frac{x - \chi_1(\tau)}{2\sqrt{a^2(t-\tau)}} \right)$$

with

$$\alpha_0 = +\infty \quad \text{for } x > \chi_1(t) ,$$
$$\alpha_0 = 0 \quad \text{for } x = \chi_1(t) = x_0 ,$$
$$\alpha_0 = -\infty \quad \text{for } x < \chi_1(t) .$$

By passage to the limit as $x \to x_0 \pm 0$, we obtain

$$[W^0(x_0 \pm 0, t) - W^0(x_0, t)] - [\tilde{V}(x_0 \pm 0, t) - \tilde{V}(x_0, t)]$$
$$= \mu_0 \frac{2}{\sqrt{\pi}} \int_0^{\pm\infty} e^{-\alpha^2} \, d\alpha = \pm \mu_0 .$$

Because of the continuity of $\tilde{V}$, there holds

$$\tilde{V}(x_0 \pm 0, t) - \tilde{V}(x_0, t) = 0 .$$

Thus

$$W^0(x_0 \pm 0, t) = W^0(x_0, t) \pm \mu_0 .$$

If $\mu(t)$ is not constant, then

$$W(x, t) = W^0(x, t) - \psi(x, t)$$

with

$$\phi(x, t) = \frac{a^2}{2\sqrt{\pi}} \int_{t_0}^{t} \frac{x - \chi_1(\tau)}{[a^2(t-\tau)]^{3/2}} e^{-[x-\chi_1(\tau)]^2/4a^2(t-\tau)} |\mu(t) - \mu(\tau)|\, d\tau .$$

Since $\mu(t)$ is assumed to be differentiable, this integral possesses the same singularity as V for $\tau = t$; it converges uniformly and is a continuous function on AP. Consequently, the limit value of $W(x, t)$ is equal to

$$W(x_0 \pm 0, t) = W(x_0, t) \pm \mu$$

as $x \to x_0 \pm 0$, which was to be proved.

We can easily see that the partial derivative $\partial V/\partial x$, like $W(x, t)$, is discontinuous for $x = x_0$. This derivative is equal to

$$\frac{\partial V}{\partial x} = -\frac{1}{2a\sqrt{\pi}} \frac{1}{2} \int_0^t \frac{x - \chi_1(\tau)}{(t-\tau)^{3/2}} e^{-[x-\chi_1(\tau)]^2/4a^2(t-\tau)} \nu(\tau)\, d\tau$$

and equals $-W(x, t)$ with density

$$\mu(t) = \frac{\nu(t)}{2} .$$

Hence, it follows that

$$\frac{\partial V}{\partial x}(x_0 \pm 0, t) = \frac{\partial V}{\partial x}(x_0, t) \mp \frac{\nu(t)}{2} ,$$

where the integral

$$\frac{\partial V}{\partial x}(x_0, t) = -\frac{1}{2a\sqrt{\pi}} \frac{1}{2} \int_0^t \frac{x_0 - \chi_1(\tau)}{(t-\tau)^{3/2}} e^{-[x_0-\chi_1(\tau)]^2/4a^2(t-\tau)} \nu(\tau)\, d\tau$$

is equal to half the sum of the right and left-sided derivatives of V at the point x_0:

$$\frac{1}{2}\left[\frac{\partial V}{\partial x}(x_0 + 0, t) + \frac{\partial V}{\partial x}(x_0 - 0, t)\right].$$

We must still point out that $V(x, t)$ itself does not possess a derivative at the point x_0.

Thus, we conclude the investigation of the potential along AP. The properties of the potential along BQ are completely analogous.

2. Solution of boundary-value problems

The heat potential represents a suitable analytic means for the solution of boundary-value problems.

We begin, with the first boundary-value problem for the one-sided bounded region $x \geqq \chi(t)$:

Determine the solution of the equation

$$u_t = a^2 u_{xx} \quad \text{for } x \geqq \chi_1(t),\ t \geqq t_0,$$

which satisfies the conditions

$$u(x, t_0) = \varphi(x), \qquad x > \chi_1(t_0),$$
$$u(\chi_1(t), t) = \mu(t), \qquad t > t_0.$$

Without loss of generality we can assume that $\varphi(x) = 0$. Therefore, if we form the difference between $u(x, t)$ and an arbitrary solution of the heat-conduction equation $v(x, t)$ which satisfies the same initial condition, we obtain a new function for which $\varphi(x) = 0$, while the boundary values are as before.

Under the assumption that the reduction to the homogeneous initial condition $\varphi(x) = 0$ has already been accomplished, we represent the solution in the form

$$u(x, t) = \frac{1}{2a^2} W(x, t) = \int_0^t \frac{\partial G_0}{\partial \xi}(x, t, \chi_1(\tau), \tau)\bar{\mu}(\tau)\, d\tau$$
$$= \frac{1}{4\sqrt{\pi}} \int_0^t \frac{x - \chi_1(t)}{[a^2(t-\tau)]^{3/2}} e^{-[x-\chi_1(\tau)]^2/4a^2(t-\tau)} \bar{\mu}(\tau)\, d\tau.$$

This function satisfies the equation for $x > \chi_1(t)$, is bounded at infinity, and satisfies the homogeneous initial condition for an arbitrary choice of $\bar{\mu}(t)$. For $x = \chi_1(t)$ it is discontinuous and its limit value for $x = \chi_1(t) - 0$ must be equal to $\mu(t)$:

$$\frac{\bar{\mu}(t)}{2a^2} + \frac{1}{4\sqrt{\pi}} \int_0^t \frac{\chi_1(t) - \chi_1(\tau)}{[a^2(t-\tau)]^{3/2}} e^{-[\chi_1(t) - \chi_1(\tau)]^2/4a^2(t-\tau)} \bar{\mu}(\tau)\, d\tau = \mu(t).$$

This expression is a Volterra integral equation of the second kind for the determination of the function $\bar{\mu}(\tau)$ which defines the solution $u(x, t)$. The existence of a solution follows from the general theory when $x = \chi_1(t)$ is a differentiable function.

This equation is especially simple when the boundary of the region under consideration is independent of time: $\chi_1(t) = x_0$. In this case, the integral occurring in the equation is equal to zero, and there remains

$$\bar{\mu}(t) = 2a^2 \mu(t),$$

so that the desired solution then has the form

$$u(x, t) = \frac{1}{2a\sqrt{\pi}} \int_0^t \frac{x - x_0}{(t-\tau)^{3/2}} e^{-(x-x_0)^2/4a^2(t-\tau)} \mu(\tau)\, d\tau.$$

We have already encountered this formula twice before (see Sections 3-3 and 6-3.3). However, here it provides the proof that this function indeed satisfies the differential equation and the auxiliary conditions.

The second and third boundary-value problems are solved analogously by means of the potential. We shall now consider the boundary-value problem for a bounded region for which the auxiliary conditions have the form

$$u(x, 0) = \varphi(x), \qquad \chi_1(0) < x < \chi_2(0),$$
$$u(\chi_1(t), t) = \mu_1(t), \qquad u(\chi_2(t), t) = \mu_2(t), \quad (t > 0).$$

We assume that this problem has already been reduced to the corresponding problem with homogeneous initial values ($\varphi(x) = 0$). We can then represent the solution in the form

$$u(x, t) = \frac{1}{2a^2}(W_1 + W_2) = \int_0^t \frac{\partial G_0}{\partial \xi}(x, t, \chi_1(\tau), \tau)\bar{\mu}_1(t)\, d\tau + \int_0^t \frac{\partial G_0}{\partial \xi}(x, t, \chi_2(\tau), \tau)\bar{\mu}_2(\tau)\, d\tau .$$

This function already satisfies the differential equation and the homogeneous initial condition as well as the functions $\bar{\mu}_1(t)$ and $\bar{\mu}_2(t)$ selected. It is discontinuous for $x = \chi_1(t)$ and $x = \chi_2(t)$. Its limit value for $x = \chi_1(t) + 0$ and $x = \chi_2(t) - 0$ must be equal to $\mu_1(t)$ and $\mu_2(t)$. These considerations lead to the system of integral equations

$$\frac{\bar{\mu}_1(t)}{2a^2} + \frac{1}{4\sqrt{\pi}} \int_0^t \frac{\chi_1(t) - \chi_1(\tau)}{[a^2(t-\tau)]^{3/2}} e^{-[\chi_1(t) - \chi_1(\tau)]^2/4a^2(t-\tau)} \bar{\mu}_1(\tau)\, d\tau + \frac{1}{4\sqrt{\pi}} \int_0^t \frac{\chi_1(t) - \chi_2(\tau)}{[a^2(t-\tau)]^{3/2}} e^{-[\chi_1(t) - \chi_2(\tau)]^2/4a^2(t-\tau)} \bar{\mu}_2(\tau)\, d\tau = \mu_1(t) ,$$

$$-\frac{\bar{\mu}_2(t)}{2a^2} + \frac{1}{4\sqrt{\pi}} \int_0^t \frac{\chi_2(t) - \chi_1(\tau)}{[a^2(t-\tau)]^{3/2}} e^{-[\chi_2(t) - \chi_1(\tau)]^2/4a^2(t-\tau)} \bar{\mu}_1(\tau)\, d\tau + \frac{1}{4\sqrt{\pi}} \int_0^t \frac{\chi_2(t) - \chi_2(\tau)}{[a^2(t-\tau)]^{3/2}} e^{-[\chi_2(t) - \chi_2(\tau)]^2/4a^2(t-\tau)} \bar{\mu}_2(\tau)\, d\tau = \mu_2(t) ,$$

a system of two Volterra integral equations which indeed possesses a solution.

Problems

1. Let a sphere of radius R_0 be filled with a gas of concentration u_0 at the initial moment. Let the concentration be equal to zero outside of the sphere. Find that function u which describes the diffusion process of the gases into the unbounded space. Solve the same problem for the half-space in the presence of a gas-permeable wall $z = 0$.

2. Treat the problem of heating a sphere of radius R_0, if the initial temperature is equal to zero, while the surface of the sphere maintains a constant temperature u_0.

3. Determine the temperature of a sphere on whose surface a heat exchange occurs with the environment. Let the temperature of the environment be equal to zero and the initial temperature of the sphere be equal to u_0.

4. Let a homogeneous fixed body be bounded by two concentric spheres with radii a and $2a$. Let the interior surface of the body be heat-insulated, while a heat exchange with the environment occurs on the exterior surface. Let the temperature of the environment be equal to zero. Find the temperature distribution in the body at time t, when the initial temperature of the body is equal to u_0.

5. Derive the diffusion equation for a medium which moves with constant velocity. Give an expression for Green's function in an unbounded space.

6. Consider a stationary diffusion process in a moving medium, where the velocity is regarded as constant and the diffusion along the direction of motion of the body can be neglected. Find Green's function for the half-space under the assumption that the plane $z = 0$ is impermeable to gas.

7. Construct Green's function for the layer which is bounded by the planes $z = 0$ and $z = l$, as well as Green's function for a wedge with aperture angle π/n (n is an integer) with homogeneous boundary conditions. Investigate the solution.

8. Construct Green's function for a momentary heat source of intensity Q which is distributed uniformly on the surface of a sphere of radius a.

9. Solve the problem of the heating of an infinitely long cylinder whose initial temperature is equal to zero, while the surface maintains a constant temperature. With the use of tables for Bessel functions, determine the temperature profile (select ten points on the radius) and the average temperature of the cross section for large time. Give the corresponding graphical representations.

10. Investigate the magnetizing of an infinitely long cylinder by a constant magnetic field which is parallel to the axis of the cylinder. With the use of tables for the Bessel functions, calculate the magnitude of the induction current through the cylinder cross section.

11. Construct Green's function for an infinite cylindrical region of arbitrary cross section with boundary conditions of the first kind. Consider the special case of a circular cross section. *Hint*: Represent the solution in the form

$$u(M, z, t) = \sum_{n=1}^{\infty} u_n(z, t)\phi_n(M) ,$$

where $\phi_n(M)$ are the eigenfunctions of the cylindrical cross section.

6-5. APPLICATIONS OF CHAPTER 6

1. Diffusion of a cloud

In the following section we shall investigate the diffusion of a gas cloud after it is released by the explosion of a grenade.

In the explosion of a grenade a fixed amount Q of smoke occurs which propagates to all sides and leads to the formation of a cloud. The cloud increases initially, then clears on the boundaries, its darker opaque part decreasing until the entire cloud has cleared; it then begins to disintegrate and finally disappears entirely. This form is especially discernible on clear days when the blue sky is present as a background.

The process of the propagation of a smoke cloud can be treated as the diffusion of smoke from an instantaneous point source of intensity Q in an unbounded space. Such a diffusion process is not of molecular, but rather of turbulent character; it corresponds to a certain effective coefficient D of a turbulent diffusion. We shall disregard the initial propagation of the smoke. Furthermore, we can disregard the almost entirely unessential influence of the earth. With these assumptions, the concentration of the smoke is given by

$$u(x, y, z, t) = Q\left(\frac{1}{2\sqrt{\pi Dt}}\right)^3 e^{-(x^2+y^2+z^2)/4Dt} , \qquad (D = a^2) ,$$

if the origin of coordinates is set at the place of the explosion of the grenade.

We shall now examine the question of the visibility of the cloud. The time duration during which the cloud is completely disintegrated depends on the absorption of light in the atmosphere and on the sensitivity of the apparatus of observation (the eyes, photographic plates, etc.).

As is known, the intensity of light which passes through a homogeneous layer of gas is approximately equal to

$$I = I_0 e^{-\alpha l} ,$$

when I_0 is the initial intensity of the light, α is the coefficient of absorption, and l is the thickness of the layer. The absorption coefficient α is proportional to the concentration of the absorbed gases, i.e., $\alpha = \alpha_0 u$, $\alpha_0 = \text{const}$, where u is the concentration of the gases.

If two layers of thickness l_1 and l_2, respectively, exist with different gas concentrations u_1 and u_2, we obtain

$$I = I_0 e^{-\alpha_0 u_1 l_1} e^{-\alpha_0 u_2 l_2} = I_0 e^{-\alpha_0 (u_1 l_1 + u_2 l_2)} .$$

Thus it is evident that the intensity of light which passes through a cloud whose smoke concentration changes continuously obeys the formula

$$I = I_0 e^{-\alpha_0 \int u\, dl} .$$

The visibility of the cloud is determined by the ratio I/I_0 which depends on $\int u\, dl$.

Let δ be the sensitivity limit of the observing apparatus; then the cloud is visible for

$$\frac{I_0 - I}{I_0} < \delta \qquad \text{or} \qquad \frac{I}{I_0} > 1 - \delta$$

and is completely opaque for

$$\frac{I_0 - I}{I_0} > 1 - \delta \qquad \text{or} \qquad \frac{I}{I_0} < \delta .$$

If

$$\delta < \frac{I}{I_0} < 1 - \delta ,$$

the cloud appears partially transparent to the observer. The degree of transparency depends on the magnitude of the ratio

$$\frac{I}{I_0} = e^{-\alpha_0 \int u\, dl} ,$$

that is, on the value of the integral $\int u\, dl$.

Now we place the z-axis in the direction of sight and assume that the observer is at infinity. In so doing, the cloud is projected on the x, y-plane. For an estimation of the visibility of different cloud particles which correspond to the points (x, y), we calculate

$$\int u\, dl = \int_{-\infty}^{\infty} u(x, y, z, t)\, dz = Q \left(\frac{1}{2\sqrt{\pi D t}} \right)^3 \int_{-\infty}^{\infty} e^{-(x^2 + y^2 + z^2)/4Dt}\, dz$$

$$= Q \left(\frac{1}{2\sqrt{\pi D t}} \right)^2 e^{-(x^2 + y^2)/4Dt} .$$

If the amount of smoke in the direction of sight is small

$$\int u\,dl < \frac{\delta}{\alpha_0},$$

then $I/I_0 > 1 - \delta$ and the corresponding cloud particles are completely transparent. By contrast, if the amount of smoke in the direction of observation is large

$$\int u\,dl > \frac{A}{\alpha_0},$$

then $I/I_0 < e^{-A} = \delta$ holds, that is, for a corresponding choice of $A = \ln 1/\delta$, the corresponding particles are completely opaque. For

$$\frac{\delta}{\alpha_0} \leqq \int u\,dl < \frac{A}{\alpha_0},$$

the condition

$$\alpha_0 \int u\,dl = \delta \quad \text{or} \quad \alpha_0 Q\left(\frac{1}{2\sqrt{\pi Dt}}\right)^2 e^{-\rho^2/4Dt} = \delta, \qquad (\rho^2 = x^2 + y^2)$$

defines the limit of visibility outside of which the cloud is invisible. The radius of the cloud is obviously equal to

$$\rho = 2\sqrt{-Dt \ln \frac{\delta 4\pi Dt}{Q\alpha_0}}$$

(Figure 87). For small t values the radius ρ of the cloud is small. It increases

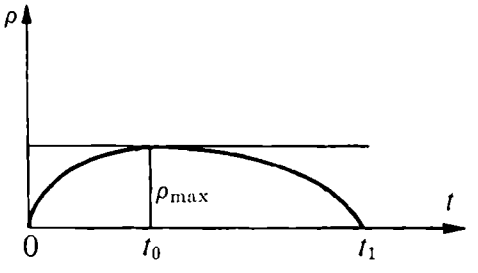


FIG. 87

with t. For

$$t = t_0 = \frac{\alpha_0 Q}{4\pi e\delta D},$$

ρ assumes the maximum

$$\rho_{max} = 2\sqrt{Dt_0} = \sqrt{\frac{\alpha_0 Q}{\pi e\delta}},$$

for $t > t_0$, ρ is smaller; and for

$$t_1 = \frac{Q\alpha_0}{\delta 4\pi D},$$

it is equal to zero (the cloud has disappeared).

By recording the disintegration of the cloud, the coefficient D of turbulent diffusion in the free atmosphere can be determined (for example, from the formula for t_1 or for t_0).

2. The demagnetization of cylinders

We shall treat the problem of demagnetization of a cylinder wound with wire. This problem arises in connection with the theory of ballistic galvanometer.[16]

By switching a magnetic field on or off, an induction current is produced in the cylinder covering. For an accurate treatment of this problem we must naturally consider the reverse action of this current on the field inside the cylinder. Ordinarily, however, this inhibiting action of the covering is not considered and the problem is solved under simplified boundary conditions.

Above all, we shall limit ourselves to this simple representation of the problem. Moreover, we consider an infinitely long cylinder of radius R, about whose surface is wound a conducting wire. The cylinder is found in a homogeneous magnetic field H_0, which is parallel to the axis of the cylinder. The axis of the cylinder coincides with the z-axis. At time $t = 0$, the field is disconnected.

Obviously, the equation

$$\Delta H = \frac{1}{a^2}\frac{\partial H}{\partial t}\,, \qquad (H = H_z) \tag{6-5.1}$$

holds in the interior of the cylinder. Here

$$a^2 = \frac{c^2}{4\pi\mu\sigma}\,.$$

Because of the axial symmetry of the field, we have

$$H_z = H(r, t)\,,$$

so that Eq. (6-5.1) assumes the form

$$\frac{\partial^2 H}{\partial r^2} + \frac{1}{r}\frac{\partial H}{\partial r} = \frac{1}{a^2}\frac{\partial H}{\partial t}\,. \tag{6-5.1'}$$

If the influence of the induction current produced in the covering by the demagnetization of the cylinder is neglected, then the boundary condition on the surface of the cylinder takes the form

$$H(R, t) = 0\,, \qquad (t > 0)\,. \tag{6-5.2}$$

The solution of Eq. (6-5.1') with the boundary condition (6-5.2) is obtained by separation of variables:

$$H(r, t) = \sum_{k=1}^{\infty} \frac{2H_0}{\mu_k^{(0)} J_1(\mu_k^{(0)})} e^{-(\mu_k^{(0)}/R)^2 a^2 t} J_0\left(\mu_k^{(0)}\frac{r}{R}\right). \tag{6-5.3}$$

Here J_0 and J_1 are Bessel functions of zero and first order, and $\mu_k^{(0)}$ is the

[16] B. A. Vvedenskii, *Zhurnal Russkovo fiziko-khimicheskovo obshchestva* **55**, 1, 1923.

kth root of the equation

$$J_0(\mu) = 0 . \tag{6-5.4}$$

Since a^2 is very small, then for sufficiently large t, we can limit ourselves to the first term in formula (6-5.3):

$$H(r, t) \approx 1.60 \cdot H_0 e^{-5.77(a^2/R^2)t} J_0\left(2.4\frac{r}{R}\right) . \tag{6-5.5}$$

From this we obtain for the flux of induction

$$\Phi(t) = 2\pi\int_0^R \mu H(r, t) r\, dr = \frac{4}{\mu_1^2}\Phi_0 e^{-\mu_1^2(a^2/R^2)t} , \tag{6-5.6}$$

where Φ_0 is the initial flux (at $t = 0$).

Formula (6-5.6) is used for practical calculations of measurements by means of a ballistic galvanometer.

For the determination of the region of applicability of (6-5.6) we have to solve the problem described above without neglecting the induction current arising in the covering on the magnetic field.

The electromotive force in the covering, as is known, is equal to

$$E_{\text{Ind}} = \oint_L E_l\, dl .$$

By using the second Maxwell equation and Eq. (6-5.1), the line integral can be transformed as follows:

$$E_{\text{Ind}} = \int_S\int \text{rot}_z\, \mathbf{c}\, dS = -\frac{\mu}{c}\int_S\int \frac{\partial H}{\partial t}\, dS = -\frac{c}{4\pi\sigma}\int_S\int \Delta H\, dS = -\frac{c}{4\pi\sigma}\oint_L \frac{\partial H}{\partial \nu}\, dl$$

or

$$E_{\text{Ind}} = -\frac{cR}{2\sigma}\frac{dH(R)}{d\nu} . \tag{6-5.7}$$

Here S is the cross section of the cylinder, L is the bounding curve, and ν is the normal to L.

The boundary condition on the surface of the cylinder reads

$$H_{z_1} - H_{z_2} = \frac{4\pi}{c} nJ ,$$

where J is the induction current in the covering and n is the number of windings per unit of length. By using $H_{z_2} = 0$ and

$$J = \frac{E_{\text{Ind}}}{\rho l}$$

(ρ is the linear resistance of the windings: l is the length of a winding), we obtain

$$H_{z_1} = H(R, t) = \frac{4\pi}{c} n \frac{E_{\text{Ind}}}{\rho l} . \tag{6-5.8}$$

By comparison of expressions (6-5.7) and (6-5.8), we arrive finally at the boundary condition

$$H(R, t) + \frac{n}{\rho\sigma} H_r(R, t) = 0 .$$

Therefore, the following equation

$$H_{rr} + \frac{1}{r} H_r = \frac{1}{a^2} H_t \tag{6-5.9}$$

with the auxiliary conditions

$$H(r, 0) = H_0 ,$$

$$H(R, t) + \alpha H_r(R, t) = 0 , \qquad \left(\alpha = \frac{n}{\rho\sigma}\right)$$

is to be solved. We try to determine the solution by separation of variables and therefore set

$$H(r, t) = X(r)T(t) .$$

The functions $X(r)$ and $T(t)$ satisfy

$$X'' + \frac{1}{r} X' + \lambda^2 X = 0 , \tag{6-5.10}$$

$$X(R) + \alpha X'(R) = 0 , \qquad (X(0) < \infty) ,$$

$$T' + \lambda^2 a^2 T = 0 , \tag{6-5.11}$$

respectively, where λ^2 is the separation parameter.

From (6-5.11) we immediately find

$$T(t) = e^{-a^2\lambda^2 t} .$$

Particular solutions of Eq. (6-5.10) are $J_0(\lambda r)$ and $N_0(\lambda r)$ (see Appendix); however, only $J_0(\lambda_r)$ is bounded for $r = 0$. Thus

$$X(r) = AJ_0(\lambda r) .$$

The boundary condition for $r = R$ for the determination of the eigenvalues yields the equation

$$J_0(\lambda R) + \alpha \frac{dJ_0(\lambda R)}{dR} = 0$$

or

$$J_0(y) - \beta y J_1(y) = 0 ,$$

where

$$\beta = \frac{\alpha}{R} , \qquad y = \lambda R . \tag{6-5.12}$$

The roots of this equation can be determined either graphically or by expansion of the Bessel function in powers of $y = \lambda R$.

Now let y_k be the roots of (6-5.12) so that

$$\lambda_k = \frac{y_k}{R} .$$

The general solution of our problem then assumes the form

$$H(r, t) = \sum_{k=1}^{\infty} A_k J_0\left(y_k \frac{r}{R}\right) e^{-y_k^2 a^2 t / R^2} . \qquad (6\text{-}5.13)$$

The coefficients A_k can be calculated from the initial condition

$$A_k = \frac{\int_0^R H_0 J_0\left(r \frac{y_k}{R}\right) r\, dr}{\int_0^R J_0^2\left(r \frac{y_k}{R}\right) r\, dr} = \frac{2H_0 J_1(y_k)}{y_k[J_0^2(y_k) + J_1^2(y_k)]} . \qquad (6\text{-}5.14)$$

The members of the series (6-5.13) decrease very rapidly since $a^2 = c^2/4\pi\rho\omega$ is large ($\approx 10^{13}$ to 10^{14}). Therefore, we can limit ourselves to the first term with sufficient accuracy:

$$H(r, t) \approx \frac{2H_0 J_1(y_1) J_0\left(y_1 \frac{r}{R}\right)}{y_1[J_0^2(y_1) + J_1^2(y_1)]} e^{-y_1^2 a^2 / R^2 t} . \qquad (6\text{-}5.15)$$

This result leads to the following expression for the flux of induction:

$$\Phi(t) \approx \Phi_0 \frac{4J_1^2(y_1)}{y_1^2[J_0^2(y_1) + J_1^2(y_1)]} e^{-y_1^2 a^2 / R^2 t} . \qquad (6\text{-}5.16)$$

Here

$$\Phi_0 = NH_0\pi R^2 ,$$

if N denotes the total number of windings.

Numerical calculation leads to the following formula for the flux of induction for different values of the parameter β:

$$\Phi(t) = \begin{cases} 0.804\Phi_0 e^{-4.75\theta} & \text{for } \beta = 0.1 , \\ 0.872\Phi_0 e^{-3.96\theta} & \text{for } \beta = 0.2 , \\ 0.912\Phi_0 e^{-3.35\theta} & \text{for } \beta = 0.3 , \end{cases} \qquad \theta = \frac{a^2}{R^2} t .$$

These formulas permit us to follow the inhibiting action of the current induced in the winding on the fading of the field in this cylinder. As β increases, that is, as the current in the covering increases, the rate of decrease of flux is small. For $\beta = 0$, expression (6-5.6) is obtained for the flux of induction. Therefore, this is the *zero*th order approximation.

In the theory of ballistic galvanometers we must know the time τ in which the flux of induction Φ_0 decreases to values determined by the sensitivity of the galvanometer which characterizes the inertia of the instrument. Let γ be the relative sensitivity of the galvanometer, that is, the galvanometer can register only the value $\Phi \geq \gamma\Phi_0$. Then τ can be easily determined by setting $\Phi = \gamma\Phi_0$ at time $t = \tau$ in

$$\Phi(t) = a\Phi_0 e^{-bt} .$$

By comparing the solution (6-5.16) with the approximate solution (6-5.6), we see that the coefficient in formula (6-5.6) is larger. This indicates a larger value for the sensitivity of the instrument for one and the same value of τ.

The following table contains the values of τ for different β, in particular for $\beta = 0$ and $\gamma = 10^{-3}$.

	R	a^2/R^2	$\tau(\beta = 0)$	$\tau(\beta = 0.1)$	$\tau(\beta = 0.2)$	$\tau(\beta = 0.3)$
Iron $\mu = 500$ $\sigma = 10^5(1/\Omega\,\mathrm{cm})$	1 cm	1.59	0.71	0.888	1.15	1.28
Alfesi[17] $\mu = 2000$ $\sigma = 1.3 \cdot 10^4$	1 cm	5.12	0.217	0.272	0.330	0.405

3. The difference method for the heat-conduction equation

For the determination of the approximate solutions of the heat-conduction equation, we often use the difference method. In the following discussion, we shall treat the first boundary problem of the heat-conduction equation.

$$\frac{\partial u}{\partial t} = a^2 \frac{\partial^2 u}{\partial x^2}, \qquad (0 < x < l, \quad 0 < t < T), \tag{6-5.17}$$

$$u(x, 0) = \varphi(x), \tag{6-5.18}$$

$$u(0, t) = \mu_1(t),$$

$$u(l, t) = \mu_2(t). \tag{6-5.19}$$

In order to convert the differential equation (6-5.17) into a system of difference equations, we first divide the interval $(0, l)$ by the points

$$x_0 = 0, \qquad x_i = ih, \ \ldots, \ x_n = l, \qquad \left(h = \frac{l}{n};\ i = 1, \cdots, n\right)$$

into n equal subintervals. We then replace the differential quotients with respect to x by two difference quotients, so that we obtain the system of difference equations

$$\frac{du_i}{dt} = a^2 \frac{u_{i+1} + u_{i-1} - 2u_i}{h^2}, \tag{6-5.17'}$$

$$u_i(0) = \varphi(x_i), \tag{6-5.18'}$$

$$u_0(t) = \mu_1(t),$$

$$u_n(t) = \mu_2(t), \tag{6-5.19'}$$

for the functions $u_i(t)$. This is a system of $n - 1$ ordinary differential equations of the first order for the determination of the $n - 1$ functions u_i. The solution of this system is uniquely determined by the initial conditions.

Now if we solve the system (6-5.17') to (6-5.19') numerically by any method, we then have a system of functions $u_i(t)$ which can be regarded as the approximate value of the function $u(x, t)$, that is, the solution of the heat-conduction equation at the points x_i. Not only the approximate value of the

[17] Aluminum-Iron-Silicon alloy.

function $u(x, t)$ at the points $x \neq x_i$, but also the approximate value of the derivatives can be calculated by interpolation.

In order to justify our method, we must show that the functions $u_i(t)$ approximate the value of the function $u(x,t)$ with arbitrary accuracy for sufficiently small h.

Theorem. The solution of the first boundary-value problem of the heat-conduction equation, with continuous initial and boundary conditions ($\varphi(0) = \mu_1(0)$, $\varphi(l) = \mu_2(0)$) can be approximated with arbitrary accuracy by solving the system (6-5.17′); that is, for any $\varepsilon > 0$, there can be determined an $h(\varepsilon)$ such that for all i

$$|u(x, t) - u_i(t)| < \varepsilon ,$$

provided we choose $h < h(\varepsilon)$ in system (6 5.17′).

First we prove several lemmas:

Lemma 1. If the function $u(x, t)$ is a solution of the first boundary-value problem of the heat-conduction equation, which including its second derivative is continuous with respect to x in the closed region $0 \leq x \leq l$, $0 \leq t \leq T$, then the functions $u_i(t)$ $u(x_i, t)$ satisfies the system of equations

$$\frac{d\bar{u}_i}{dt} = a^2 \frac{\bar{u}_{i+1} + u_{i-1} - 2\bar{u}_i}{h^2} = Q_i(t) . \tag{6-5.20}$$

Here, the $Q_i(t)$ are uniformly bounded by a number $Q_0(h)$, where $Q_0(h)$ approaches zero as $h \to 0$.

Proof. To prove the lemma we put $u(x, t)$ into the system (6-5.17′).

By using the equations

$$u_{i+1}(t) \quad u(x_i + h, t) = u(x_i, t) + hu_x(x_i, t) + \frac{h^2}{2} u_{xx}(x_i + \theta_1 h, t) , \qquad (0 \leq \theta_1 \leq 1) ,$$

$$\bar{u}_{i-1}(t) = u(x_i - h, t) = u(x_i, t) \quad hu_x(x_i, t) + \frac{h^2}{2} u_{xx}(x_i - \theta_2 h, t) , \qquad (0 \leq \theta_2 \leq 1)$$

and also Eq. (6-5.17), we then obtain

$$Q_i(t) = \frac{a^2}{2} [u_{xx}(x_i + \theta_1 h, t) + u_{xx}(x_i - \theta_2 h, t) - 2u_{xx}(x_i, t)] . \tag{6 5.21}$$

Since the region under consideration is closed, the uniform continuity of $u_{xx}(x, t)$ in this region follows from the continuity of $u_{xx}(x, t)$, that is, for any $\eta > 0$, there exists an $h(\eta)$ such that

$$|u_{xx}(x + \theta h, t) - u_{xx}(x, t)| < \eta$$

holds for arbitrary x and t. From this, however,

$$|Q_i(t)| < Q_0 \quad a^2\eta$$

follows for $h < h(\eta)$, whereby the lemma is proved.

Note. If the function $u(x, t)$ has four continuous derivatives, then

$$Q_0 \leq \frac{a^2 h^2}{12} U^{IV} , \tag{6 5.22}$$

holds, where U^{IV} is the maximum of $u^{IV}(x, t)$ in the region $0 \leqq x \leqq l$, $0 \leqq t \leqq T$. This result follows from the equations

$$u(x_i + h, t) = u(x_i, t) + hu_x(x_i, t) + \frac{h^2}{2}u_{xx}(x_i, t) + \frac{h^3}{3!}u_{xxx}(x_i, t) + \frac{h^4}{4!}u^{IV}(x_i + \theta_1 h, t),$$

$$u(x_i - h, t) = u(x_i, t) - hu_x(x_i, t) + \frac{h^2}{2}u_{xx}(x_i, t) - \frac{h^3}{3!}u_{xxx}(x_i, t) + \frac{h^4}{4!}u^{IV}(x_i - \theta_2 h, t).$$

Lemma 2. The minimum of the solution of the system of equations

$$\frac{dw_i}{dt} - a^2\frac{w_{i+1} + w_{i-1} - 2w_i}{h^2} = Q_i(t) \quad \text{for } Q_i(t) \geqq 0 \tag{6-5.23}$$

cannot be smaller than the minimum of the initial value or the minimum of the functions $w_0(t)$ and $w_n(t)$.

Proof. The minimum of the set of functions $w_i(t)$ for $i = i_0$ must be attained for $t = t_0$ ($t_0 > 0$), that is, $w_i(t) \geqq w_{i_0}(t_0)$.

We consider the equation for w_{i_0} at $t = t_0$. Obviously,

$$\left.\frac{dw_{i_0}}{dt}\right|_{t=t_0} \leqq 0$$

holds (the "$<$" sign can only hold for $t_0 = T$), and

$$a^2\frac{w_{i_0+1} + w_{i_0-1} - 2w_{i_0}}{h^2} \geqq 0.$$

This conclusion, however, is a contradiction, in the case $Q_{i_0}(t_0) > 0$. However, if $Q_{i_0}(t_0) = 0$, then $w_{i_0+1}(t_0) = w_{i_0-1}(t_0) = w_{i_0}(t_0)$.

If we continue this argument for $i_0 + 1$ and so forth, we arrive at a contradiction when $Q_{i_0+k}(t_0)$ is positive for one k. If, however,

$$Q_{i_0+k}(t_0) = 0,$$

then we would necessarily have

$$w_{i_0}(t_0) = w_{i_0+1}(t_0) = \cdots = w_{i_0+k}(t_0) = \cdots = w_n(t_0),$$

from which it would follow that the minimum of the set of functions $w_i(t)$ ($i = 1, \ldots, n-1$) cannot be smaller than the minimum of the function $w_n(t)$. A corresponding lemma holds for the maximum of the solution of the system (6-5.23) when

$$Q_i(t) \leqq 0.$$

Conclusion 1. If the set of functions $\{\bar{v}_i(t)\}$ ($i = 1, \ldots, n-1$) satisfies the system of equations

$$\frac{d\bar{v}_i}{dt} - a^2\frac{\bar{v}_{i+1} + \bar{v}_{i-1} - 2\bar{v}_i}{h^2} = \bar{Q}_i(t),$$

and the set of functions $\{\bar{\bar{v}}_i(t)\}$ $(i = 1, \ldots, n-1)$ satisfies the system of equations

$$\frac{d\bar{\bar{v}}_i}{dt} - a^2 \frac{\bar{\bar{v}}_{i+1} + \bar{\bar{v}}_{i-1} - 2\bar{\bar{v}}_i}{h^2} = \bar{\bar{Q}}_i(t) ,$$

where

$$\bar{Q}_i(t) \leqq \bar{\bar{Q}}_i(t) , \qquad \bar{v}_0(t) \leqq \bar{\bar{v}}_0(t) , \qquad \bar{v}_n(t) \leqq \bar{\bar{v}}_n(t) , \qquad \bar{v}_i(0) \leqq \bar{\bar{v}}_i(0)$$

holds, then

$$\bar{v}_i(t) \leqq \bar{\bar{v}}_i(t) , \qquad (i = 1, 2, \ldots, n-1) .$$

The proof of this follows directly from Lemma 2, when it is applied directly to the functions $w_i(t) = \bar{\bar{v}}_i(t) - v_i(t)$.

Conclusion 2. The solution of the homogeneous system (6-5.23) satisfies the condition

$$|w_i(t)| < \varepsilon ,$$

provided

$$|w_i(0)| < \varepsilon , \qquad |w_0(t)| < \varepsilon \quad \text{and} \quad |w_n(t)| < \varepsilon .$$

This assertion also results directly from Lemma 2.

Lemma 3. If the function $u(x, t)$ is a solution of the first boundary-value problem of the heat-conduction equation, which including its second derivative with respect to x is continuous in the closed region $0 \leqq x \leqq l$, $0 \leqq t \leqq T$, then it can be approximated for sufficiently small h with arbitrary accuracy by the solution of the system of difference equations (6-5.17').

Proof. For the proof we consider the difference of the functions $\bar{u}_i(t) = u(x_i, t)$ and $u_i(t)$:

$$\bar{v}_i(t) = \bar{u}_i(t) - u_i(t)$$

These functions satisfy the system of equations

$$\frac{d\bar{v}_i}{dt} - a^2 \frac{\bar{v}_{i+1} + \bar{v}_{i-1} - 2\bar{v}_i}{h^2} = Q_i(t) ,$$

$$\bar{v}_i(0) = 0 , \qquad \bar{v}_0(t) = 0 , \qquad \bar{v}_n(t) = 0 ,$$

where

$$Q_i(t) \leqq Q_0(h) , \qquad (Q_0(h) \underset{h \to 0}{\to} 0) .$$

We shall prove that

$$\bar{v}_i(t) \leqq Q_0 t , \qquad (i = 1, 2, \ldots, n-1)$$

To do so we consider the system of auxiliary functions

$$\bar{\bar{v}}_i(t) = Q_0 t \quad \text{for all } i = 0, 1, \ldots, n .$$

This system satisfies the equations

$$\frac{d\bar{\bar{v}}_i}{dt} - a^2 \frac{\bar{\bar{v}}_{i+1} - \bar{\bar{v}}_{i-1} - 2\bar{\bar{v}}_i}{h^2} = Q_0$$

with the auxiliary conditions

$$\bar{\bar{v}}_i(0) = 0 , \qquad \bar{\bar{v}}_0(t) = Q_0 t , \qquad \bar{\bar{v}}_n(t) = Q_0 t .$$

On the basis of Conclusion 1 of Lemma 2, the relation

$$\bar{v}_i(t) \leqq \bar{\bar{v}}_i(t) = Q_0 t$$

holds. Correspondingly, by consideration of the system of functions $-\bar{\bar{v}}_i(t)$, we obtain

$$\bar{v}_i(t) \geqq -\bar{\bar{v}}_i(t) = -Q_0 t .$$

Hence it follows that

$$|\bar{v}_i(t)| \leqq Q_0 t , \qquad (i = 1, 2, \ldots, n-1) , \tag{6-5.24}$$

where

$$Q_0(h) \to 0 \quad \text{for } h \to 0 ,$$

thus proving Lemma 3.

We shall now list several properties of the solution of the heat-conduction equation which will be of use in the following discussion.

Lemma 4. If the function $u(x, t)$ is a solution of the first boundary-value problem of the heat-conduction equation where the initial function $\varphi(x)$ including its derivatives up to the second order are continuous in the interval $0 \leqq x \leqq l$, and if the boundary values $\mu_1(t)$ and $\mu_2(t)$ including its derivatives of the first order are continuous and satisfy the transition conditions

$$\varphi(0) = \mu_1(0) , \qquad \varphi(l) = \mu_2(0) , \qquad \varphi''(0) = \mu_1'(0) , \qquad \varphi''(l) = \mu_2'(l) .$$

Then the function $u(x, t)$ possesses a continuous second derivative with respect to x in the closed region $0 \leqq x \leqq l$, $0 \leqq t \leqq T$.

We consider a function $z_1(x, t)$ which satisfies the heat-conduction equation and the auxiliary conditions

$$z_1(x, 0) = a^2 \varphi''(x) , \qquad z_1(0, t) = \mu_1'(t) , \qquad z_1(l, t) = \mu_2'(t) ,$$

and we form the function

$$z(x, t) = \varphi(x) + \int_0^t z_1(x, \tau)\, d\tau .$$

The initial and boundary values of the function so formed are then

$$z(x, 0) = \varphi(x) ,$$

$$z(0, t) = \mu_1(0) + \int_0^t \mu_1'(\tau)\, d\tau = \mu_1(t) , \qquad (\varphi(0) = \mu_1(0)) ,$$

$$z(l, t) = \mu_2(t) .$$

On the basis of

$$\frac{\partial z}{\partial t} = z_1(x, t) ,$$

$$\frac{\partial^2 z}{\partial x^2} = \varphi''(x) + \int_0^t \frac{\partial^2 z_1}{\partial x^2}(x, \tau)\, d\tau = \frac{1}{a^2} z_1(x, 0) + \frac{1}{a^2} \int_0^t \frac{\partial z_1}{\partial \tau}(x, \tau)\, d\tau = \frac{1}{a^2} z_1(x, t) ,$$

we recognize that the function $z(x, t)$ for $t > 0$ satisfies the heat-conduction equation. Consequently,

$$z(x, t) \equiv u(x, t) ,$$

whereby the continuity of the function $u(x, t)$ and its derivatives up to and including the second order is demonstrated.

Conclusion. The solution of the first boundary-value problem for the heat-conduction equation can be approximated for sufficiently small h by the solution of the above system of difference equations with arbitrary accuracy. In this case it is assumed that the initial function $\varphi(x)$ including its second derivative, and the boundary value together with its first derivative are continuous, and the transition conditions are satisfied.

We turn now to the proof of our principle theorem. Here it is sufficient to show that the additional differentiability requirements on the functions $\varphi(x)$, $\mu_1(t)$, and $\mu_2(t)$ in the previous conclusion of Lemma 4 are superfluous. We consider the function $u(x, t)$ which is a solution of the following first boundary-value problem with continuous initial and boundary conditions:

$$u(x, 0) = \varphi(x) , \qquad u(0, t) = \mu_1(t) , \qquad u(l, t) = \mu_2(t)$$
$$(\varphi(0) = \mu_1(0) , \qquad \varphi(l) = \mu_2(0)) .$$

Furthermore, we consider arbitrary functions $\hat{\varphi}(x)$, $\hat{\mu}_1(t)$, and $\hat{\mu}_2(t)$ with the following properties: The function $\hat{\varphi}(x)$ and its first two derivatives are continuous, while $\hat{\mu}_1(t)$ and $\hat{\mu}_2(t)$ and their first derivatives are continuous and satisfy the transition conditions

$$\hat{\varphi}(0) = \hat{\mu}_1(0) , \qquad \hat{\varphi}(l) = \hat{\mu}_2(0) ,$$
$$\hat{\varphi}''(0) = \hat{\mu}_1'(0) , \qquad \hat{\varphi}''(l) = \hat{\mu}_2'(0) .$$

Finally, these functions are to approximate the initial and boundary values of our function so that simultaneously

$$|\hat{\varphi}(x) - \varphi(x)| < \frac{\varepsilon}{3} ,$$

$$|\hat{\mu}_1(t) - \mu_1(t)| < \frac{\varepsilon}{3} ,$$

$$|\hat{\mu}_2(t) - \mu_2(t)| < \frac{\varepsilon}{3}$$

hold.[18]

Let $\hat{u}(x, t)$ be the solution of the corresponding boundary-value problem. On the basis of the principle of maximum, we have

$$|\hat{u}(x, t) - u(x, t)| < \frac{\varepsilon}{3} .$$

[18] Such functions can be constructed in many ways. For example, they can be obtained so that $\varphi(x)$, $\mu_1(t)$, $\mu_2(t)$ can be replaced by linear segments; and these change at the corner points so that the differentiability and transition conditions are satisfied.

Correspondingly, because of the principle of maximum for the difference system (Conclusion 2, Lemma 2) the inequality

$$|\hat{u}_i(t) - u_i(t)| < \frac{\varepsilon}{3}$$

holds, where $\hat{u}_i(t)$ and $u_i(t)$ are the solutions of the corresponding difference system with the auxiliary conditions

$$\hat{u}_i(0) = \hat{\varphi}(x_i)\,, \qquad \hat{u}_0(t) = \hat{\mu}_1(t)\,, \qquad \hat{u}_n(t) = \hat{\mu}_2(t)\,,$$
$$u_i(0) = \varphi(x_i)\,, \qquad u_0(t) = \mu_1(t)\,, \qquad u_n(t) = \mu_2(t)\,.$$

Because of the conclusion from Lemma 4, the expression

$$|\hat{u}(x, t) - \hat{u}_i(t)| < \frac{\varepsilon}{3}$$

holds when h is chosen sufficiently small [see formula (6-5.18)], and hence

$$Q_0 < \frac{\varepsilon}{3T}\,.$$

From the inequalities obtained, we therefore find

$$|u(x, t) - u_i(t)| \leq |u(x, t) - \hat{u}(x, t)| + |\hat{u}(x, t) - \hat{u}_i(t)| + |\hat{u}_i(t) - u_i(t)| < \varepsilon\,,$$

whereby our theorem is proved.

Now, in the system of equations (6-5.17′) we replace the derivatives with respect to t by the differences

$$\frac{u_i^{j+1} - u_i^j}{k}$$

where $k = \Delta t$ and $u_i^j = u(x_i, t_j)$. Then we obtain the system of difference equations

$$u_i^{j+1} - u_i^j = \gamma a^2(u_{i+1}^j + u_{i-1}^j - 2u_i^j)\,, \qquad \left(\gamma = \frac{k}{h^2}\right)$$

$$\left(i = 1, 2, \ldots, n-1\,; \qquad j = 0, 1, \ldots, N-1\,; \qquad N = \frac{T}{k}\right)$$

or

$$u_i^{j+1} = (1 - 2a^2\gamma)u_i^j + a^2\gamma(u_{i+1} + u_{i-1}^j)\,, \tag{6-5.17″}$$

with the auxiliary conditions

$$u_i^0 = \varphi(x_i)\,, \qquad u_0^j = \mu_1(t_j)\,, \qquad u_n^j = \mu_2(t_j)\,. \tag{6-5.18″}$$

If $\gamma < 1/2a^2$, then all coefficients in Eq. (6-5.17″) are positive, and for the difference system, the principle of maximum is valid.

Specifically, if we assume that $u_{i_0}^{j_0+1}$ is the maximum of the solution $(0 < i_0 < n;\ j_0 > 0)$ and recall that all the coefficients of Eq. (6-5.17″) are positive and that their sum is always equal to one, we can then conclude that Eq. (6-5.17′) can hold only if the relations

$$u_{i_0}^{j_0+1} = u_{i_0}^{j_0} = u_{i_0-1}^{j_0} = u_{i_0+1}^{j_0+1}$$

are satisfied.

If we continue this argument, we arrive at the conclusion that the maximum must be assumed either for $i = 0$, for $i = n$, or for $j = 0$.

By repeating the above arguments for the difference system 6-5.17′, we are easily convinced that the solution of the difference systems 6-5.17″ approximates the solution of the boundary-value problem for the heat-conduction equation with arbitrary accuracy.

The difference system (6-5.17″) arises in the numerical integration by means of Euler's method from the difference system (6-5.17′). If we use another method for the numerical integration, we obtain another type of difference system which corresponds to the heat-conduction equation.

Approximate solutions of the heat-conduction equation can also be obtained by replacing only the derivative with respect to t by differences; the determination of the approximate solution leads to solving the boundary-value problem

$$\frac{u^{j+1}(x) - u^j(x)}{k} = a^2 \frac{d^2 u^{j+1}(x)}{dx^2}$$

or

$$a^2 \frac{d^2 u^{j+1}}{dx^2} - \frac{1}{k} u^{j+1}(x) = -\frac{1}{k} u^j(x)$$

with the auxiliary conditions

$$u^0(x) = \varphi(x), \qquad u^j(0) = \mu_1(t_j), \qquad u^j(l) = \mu_2(t_j)^{19}.$$

Until now the solution of the homogeneous heat-conduction equation with constant coefficients has been considered. The inhomogeneous equation as well as linear equations of the parabolic type with variable coefficients are treated in a completely analogous way if the coefficients are assumed to be continuous and differentiable with respect to t.[20] The differentiability with respect to t is necessary if we wish to establish an analog to Lemma 4. If the coefficients of the equation depend only on x, the proof remains unchanged.

[19] Details of the solution of the heat-conduction equation according to this scheme are found in V. I. Smirnov, *Textbook of Higher Mathematics*, Part IV, VEB Deutscher Verlag der Wissenschaften, Berlin, 1958.

[20] Here it is assumed that the corresponding boundary-value problem for continuous initial and boundary conditions is solvable.

7

ELLIPTIC DIFFERENTIAL EQUATIONS (CONTINUED)

7-1. SOME FUNDAMENTAL PROBLEMS WHICH LEAD TO THE DIFFERENTIAL EQUATION $\Delta v + cv = 0$

1. Forced vibrations

Many problems which are related to forced vibrations (mechanical, acoustic, electromagnetic vibrations, etc.) lead to the so-called vibration equation

$$\Delta v + k^2 v = 0\,, \quad (k^2 = c > 0)\,. \tag{7-1.1}$$

We consider as an example a membrane S which is fixed along its boundary C and vibrates under the influence of a time periodic force. The corresponding differential equation has the form

$$\Delta_2 \bar{u} = \frac{1}{a^2}\bar{u} - F_0(x, y)\cos \omega t\,. \tag{7-1.2}$$

For the investigation of periodic processes we use suitable complex functions in which we replace Eq. (7-1.2) by the equation

$$\Delta_2 u = \frac{1}{a^2}u_{tt} - F_0(x, y)e^{i\omega t}\,. \tag{7-1.3}$$

Obviously $\bar{u}$ is then the real part of the function u in (7-1.3).

We shall now seek the forced vibrations given by

$$u = ve^{i\omega t}\,. \tag{7-1.4}$$

For the amplitude v of these forced vibrations we obtain the equation

$$\Delta_2 v + k^2 v = -F_0(x, y)\,, \quad \left(k = \frac{\omega}{c}\right), \tag{7-1.5}$$

to which we add the boundary conditions

$$v\,|_C = 0\,. \tag{7-1.6}$$

If the boundary C is not fixed, but undergoes periodic vibrations with the same frequency

$$u\,|_C = f_0 e^{i\omega t}\,, \tag{7-1.6'}$$

then the function v must satisfy the inhomogeneous boundary condition

$$v|_C = f_0 \qquad 7(-1.6'')$$

on C.

As discussed above, the problems of forced vibrations are characteristic both for accoustics as well as for the theory of electromagnetic fields. Moreover, problems of forced vibrations often occur in inhomogeneous media in particular, and sometimes in piecewise homogeneous media (for example, in a space in which subdomains of a different nature exist).

To this same category of questions belong the problems of diffraction theory; we shall treat them further below.

2. Diffusion of gases with decomposition phenomena and with chain reactions

The diffusion of certain gases (radium emanation, for example) is accompanied by a decomposition of molecules of the diffusing gases. The decomposition velocity is usually assumed to be proportional to the concentration of the gases; in the diffusion equation it is equivalent to the appearance of negative gas sources. In the case of a stationary diffusion process, the following equation holds:

$$D\Delta v + cv = 0\,, \quad (c < 0)\,. \qquad (7\text{-}1.7)$$

Here D denotes the diffusion coefficient. As was noted in Section 6-3.3, the case $c > 0$ is of special interest. This case corresponds to a diffusion process with a chain reaction which leads to an increase of the diffusing particles. In the stationary case we therefore obtain the equation

$$\Delta v + cv = 0\,, \quad (c > 0)\,,$$

since the chain reaction is equivalent to the existence of sources of the diffusing substances whose numbers are proportional to the concentration $v(x, y, z)$.

3. Diffusion in a moving medium

In Chapter 4 we treated the problem of the diffusion of gases in a fixed medium. Here we shall consider the diffusion of a gas in a given stationary stream whose velocity at the point $M(x, y, z)$ has components $\theta_1(x, y, z)$, $\theta_2(x, y, z)$, $\theta_3(x, y, z)$. The amount of gas which passes the surface element $d\sigma$ at the point $M(x, y, z)$ is equal to

$$dQ = -D\boldsymbol{n}\,\mathrm{grad}\,u\,d\sigma + u\boldsymbol{\theta}\boldsymbol{n}\,d\sigma\,,$$

where $u(x, y, z)$ is the concentration of the gases (given per units of volume), $\boldsymbol{n}$ is the unit normal vector at the element $d\sigma$, D is the diffusion coefficient at the point (x, y, z), and $\boldsymbol{\ell}(x, y, z)$ is the velocity vector of the stream.

On the basis of the law of conservation of mass, we obtain

$$\int_\Sigma [-D\boldsymbol{n}\,\mathrm{grad}\,u + u\boldsymbol{\theta}\boldsymbol{n}]\,d\sigma = 0\,.$$

for a fixed region of space T with boundary Σ. By means of the Gauss-

Ostrogradski formula these surface integrals can be transformed into volume integrals; thus we obtain

$$\int_T [\operatorname{div}(D \operatorname{grad} u) - \operatorname{div}(u\vartheta)]\, d\tau = 0 .$$

Since T is arbitrary, we obtain the diffusion equation in the given stream

$$\operatorname{div}(D \operatorname{grad} u) - \operatorname{div}(u\vartheta) = 0 , \tag{7-1.8}$$

or in scalar form

$$\frac{\partial}{\partial x}\left(D\frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial y}\left(D\frac{\partial u}{\partial y}\right) + \frac{\partial}{\partial z}\left(D\frac{\partial u}{\partial z}\right) - \frac{\partial}{\partial x}(u\vartheta_1) - \frac{\partial}{\partial y}(u\vartheta_2) - \frac{\partial}{\partial z}(u\vartheta_3) = 0 . \tag{7-1.8'}$$

From this differential equation we are also led to the problem of heat propagation in a moving medium.

Now let us consider the following example:

In the half-space $z \geqq 0$ let an air stream with constant velocity u_0 be given. The stream has the direction of the x-axis. If we take the diffusion coefficients to be constant, then from Eq. (7-1.8) we obtain the equation

$$D\Delta u - u_0 \frac{\partial u}{\partial x} = 0 .$$

Now if we set

$$u = ve^{\mu x}$$

and choose

$$\mu = \frac{u_0}{2D} ,$$

we obtain the equation

$$\Delta v + cv = 0$$

for the function $v(x, y, z)$, where

$$c = -\frac{u_0^2}{4D} < 0 .$$

4. Formulation of the interior boundary-value problems for the equation $\Delta v + cv = 0$

As we showed in Chapter 1 in the investigation of the canonical forms of partial differential equations with constant coefficients, every elliptic differential equation with constant coefficients can be brought to the form

$$\Delta v + cv = 0 . \tag{7-1.9}$$

The properties of the solutions of Eq. (7-1.9) essentially depend on the signs of the coefficient c. This fact is immediately clear from a physical standpoint if we interpret the equation somewhat as we do a diffusion equation.

We next consider the equation of the uniqueness of the solution of the first boundary-value problem for Eq. (7-1.9). For Eq. (7.19) in the case $c < 0$, the principle of the maximum reads:

A solution $v(M)$ of the equation

$$\Delta v + cv = 0 \quad (c < 0),$$

which is defined in the interior of a fixed region T with the boundary Σ, can assume neither a positive maximum nor a negative minimum value in the interior points of T.

If we assume that at a fixed point M in T the function $v(M)$ has a positive maximum $[v(M_0) > 0]$, then at the point M_0 we have

$$\frac{\partial^2 v}{\partial x^2} \leqq 0, \quad \frac{\partial^2 v}{\partial y^2} \leqq 0, \quad \frac{\partial^2 v}{\partial z^2} \leqq 0,$$

and consequently $\Delta v \leqq 0$, in contradiction to the fact that the coefficient c is negative and $v(M_0)$ is positive.[21]

The principle of the maximum automatically leads to the uniqueness of the solution of the first boundary-value problem for Eq. (7-1.9).

There can exist only one solution of the equation

$$\Delta v + cv = 0, \quad (c < 0),$$

which is defined and continuous in the closed region $T + \Sigma$ and on the boundary Σ assumes the prescribed value

$$v|_\Sigma = f.$$

Therefore, if we assume the existence of two different solutions v_1 and v_2, consider their difference $v_1 - v_2$, and carry out the arguments described above (see Chapters 3 and 4), then we obtain from this assumption a contradiction to the principle of the maximum.

For $c = 0$, we obtain the first boundary-value problem for the Laplace differential equation whose uniqueness has already been established.

If $c > 0$, then the problem need not be uniquely solvable. In Chapter 5 we considered the eigenvalue problems

$$\Delta v + \lambda v = 0, \quad v|_\Sigma = 0.$$

By examples, we recognized the existence of nontrivial solutions (eigenfunctions) for $\lambda > 0$. Obviously the question of the ambiguous or unique solvability of the first boundary-value problem is equivalent to the question of whether or not c coincides with one of the eigenvalues λ_n of the region T considered.

7-2. GREEN'S FUNCTIONS

1. Green's function

The theory of the potential developed in Chapter 4 for the Laplace differential equation can be extended to the Eq. (7-1.9). For the construction of Green's function we consider a solution v_0 dependent only on r. In polar

[21] Compare this with the proof of the principle of the maximum for the heat-conduction equation.

coordinates the Laplace differential equation for the function $v_0(r)$ reads

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dv_0}{dr}\right) = \frac{1}{r}\frac{d^2(rv_0)}{dr^2}.$$

This equation leads to the ordinary differential equation

$$\frac{d^2w}{dr^2} + cw = 0, \quad (w = v_0 r).$$

With the notation $c = k^2$ for $c > 0$ or $c = -\kappa^2$ for $c < 0$, we obtain

$$\frac{d^2w}{dr^2} + k^2w = 0, \quad (c > 0), \tag{7-2.1}$$

$$\frac{d^2w}{dr^2} - \kappa^2w = 0, \quad (c < 0). \tag{7-2.1'}$$

From Eq. (7-2.1) we find

$$w = C_1e^{ikr} + C_2e^{-ikr}, \tag{7-2.2}$$

and, correspondingly,

$$v_0 = C_1\frac{e^{ikr}}{r} + C_2\frac{e^{-ikr}}{r}. \tag{7-2.3}$$

For real k we have two linearly independent solutions e^{ikr}/r and e^{-ikr}/r, which correspond to real linearly independent solutions $(\cos kr)/r$ and $(\sin kr)/r$. For $c < 0$ $(c = -\kappa^2)$, we obtain from (7-2.1') the two linearly independent solutions

$$\frac{e^{-\kappa r}}{r} \quad \text{and} \quad \frac{e^{\kappa r}}{r}, \quad (\kappa > 0). \tag{7-2.4}$$

The functions

$$\frac{e^{\pm ikr}}{r}, \quad (c > 0) \quad \text{and} \quad \frac{e^{\pm\kappa r}}{r}, \quad (c < 0)$$

for $r = 0$ are discontinuous and approach infinity at $r = 0$ like $1/r$. A singularity of such a character is possessed by Green's function of the Laplace differential equation $(c = 0)$, which is proportional to $1/r$.

We shall investigate the behavior of this function at infinity. The case $c < 0$ corresponds to a process accompanied by absorption [compare with the diffusion equation (7-1.6)]. The solution $e^{-\kappa r}/r$ at infinity tends towards zero exponentially, which in the terminology of diffusion processes denotes a decrease in the concentration because of absorption. The larger the coefficient $|c| = \kappa^2$ which characterizes the absorption intensity, the stronger the decrease. The second solution at infinity increases exponentially and for problems in unbounded space has no physical significance (it can be interpreted in terms of sources existing at infinity).

The case $c = k^2 > 0$ corresponds to forced wave processes (see Section 7-1.1). The function v represents the amplitude of the function

$$u(M, t) = v(M)e^{i\omega t},$$

which satisfies the vibration equation (Section 7-1).

One of the principal solutions of Eq. (7-2.1),

$$v_0(r) = \frac{e^{-ikr}}{r},$$

corresponds to the vibration process

$$u_0(r, t) = \frac{e^{i(\omega t - kr)}}{r}.$$

This solution has the character of a spherical wave which proceeds from the source at the point $r = 0$. The second solution

$$v_0(r) = \frac{e^{ikr}}{r}$$

corresponds to the vibration process

$$u_0(r, t) = \frac{e^{i(\omega t - kr)}}{r},$$

which has the character of a spherical wave proceeding from infinity to the point $r = 0$. Obviously this solution has no direct physical significance for processes which are produced by point sources in unbounded space.

We still note that the function $v(M)$ can be regarded as the amplitude of a vibration of the type $e^{i\omega t}$ or $e^{-i\omega t}$. We choose the time factor $e^{i\omega t}$. If we choose $e^{-i\omega t}$, then the divergent waves take the form

$$u_0(r, t) = \frac{e^{-i(\omega t - kr)}}{r},$$

that is, this form corresponds to the second solution

$$v_0(r) = \frac{e^{ikr}}{r}.$$

The first solution

$$v_0(r) = \frac{e^{-ikr}}{r},$$

therefore, has no physical significance.

2. Integral representation of the solution

For Eq. (7-1.1) in case $c \neq 0$ we can give formulas which correspond to Green's formula for the Laplace differential equation. If we introduce the relation

$$\boldsymbol{L}(u) = \Delta u + cu, \tag{7-2.5}$$

then we obtain directly the formula

$$\int_T (u\boldsymbol{L}(v) - v\boldsymbol{L}(u))\, d\tau = \int_\Sigma \left(u\frac{\partial v}{\partial \nu} - v\frac{\partial u}{\partial \nu}\right) d\sigma \tag{7-2.6}$$

as an analog and direct sequence of Green's formula (see Section 4-2). In the

last formula, in place of v, we introduce one of the Green's functions, for example, $e^{-\kappa r}/r$. By a verbatim argument from Section 4-2, we arrive at the following analog of Green's fundamental formula:

$$u(M_0) = -\frac{1}{4\pi}\int_\Sigma \left[u\frac{\partial}{\partial \nu}\left(\frac{e^{-\kappa r}}{r}\right) - \frac{e^{-\kappa r}}{r}\frac{\partial u}{\partial \nu}\right] d\sigma_M + \frac{1}{4\pi}\int_T f(M)\frac{e^{-\kappa r}}{r}\, d\tau_M\,, \quad (r = r_{MM_0})\,, \tag{7-2.7}$$

where $u(M)$ is a solution of the inhomogeneous equation $L(u) = f(M)$.

For $c = k^2$ we obtain the formula

$$u(M_0) = -\frac{1}{4\pi}\int_\Sigma \left[u\frac{\partial}{\partial \nu}\left(\frac{e^{-ikr}}{r}\right) - \frac{e^{-ikr}}{r}\frac{\partial u}{\partial \nu}\right] d\sigma + \frac{1}{4\pi}\int_T f(M)\frac{e^{-ikr}}{r}\, d\tau_M\,, \tag{7-2.7'}$$

which we derived in Chapter 5 as a consequence of the Kirchhoff wave formula.

We now introduce the concept of Green's function of the equation $L(u) = 0$ for a given region T with boundary Σ. Let $v(M)$ be a regular solution of the equation $L(v) = 0$ everywhere in T. Then from formula (7-2.6) we find

$$0 = -\int_\Sigma \left(u\frac{\partial v}{\partial \nu} - v\frac{\partial u}{\partial \nu}\right) d\sigma + \int_T fv\, d\tau\,. \tag{7-2.8}$$

By comparing Eq. (7-2.8) with Eq. (7-2.7), we obtain

$$u(M_0) = -\int_\Sigma \left[u\frac{\partial}{\partial \nu}\left(\frac{e^{-\kappa r}}{4\pi r} + v\right) - \left(\frac{e^{-\kappa r}}{4\pi r} + v\right)\frac{\partial u}{\partial \nu}\right] d\sigma_M + \int_T \left(\frac{e^{-\kappa r}}{4\pi r} + v\right) f(M)\, d\tau_M\,, \quad (r = r_{MM_0})\,. \tag{7-2.9}$$

This formula holds for each regular solution $v(M)$ of the equation $\Delta v - \kappa^2 v = 0$ in T. Since v can be chosen arbitrarily, then the following necessarily holds:

$$u(M_0) = -\int_\Sigma u(M)\frac{\partial G(M_0 M)}{\partial \nu}\, d\sigma_M + \int_T G(M_0, M) f(M)\, d\tau_M \tag{7-2.10}$$

Here

$$G(M_0, M) = \frac{e^{-\kappa r}}{4\pi r} + v \tag{7-2.11}$$

is the Green's function with the following properties:

1. $G(M, M_0)$ for $M = M_0$ is infinite like $1/(4\pi r)$, which follows from formula (7-2.11).

2. $G(M, M_0)$ satisfies the equation $L(u) = 0$ everywhere in T except at the point M_0,

3. $G(P, M_0) = 0$ for a point P which lies on the boundary Σ.

The question of the existence of Green's function is related to the question of whether a function v exists which satisfies the equation

$$L(v) = 0 \quad \text{in } T$$

and the boundary condition

$$u = \frac{e^{\kappa r}}{4\pi r} \quad \text{on } \Sigma .$$

Obviously the function $G(M, M_0)$ is uniquely defined for each region which allows the unique solvability of the first boundary-value problem. In particular, this function is defined for $c = -\kappa^2 < 0$ for each region. In the simplest case, a direct representation of Green's function can be given. For this we use methods which are analogous to the method of images in electrostatics.[22]

Thus, for example, Green's function for the half-space $z > 0$ has the form

$$G(M, M_0) = \frac{e^{\kappa r}}{4\pi r} - \frac{e^{\kappa r_1}}{4\pi r_1} ,$$

$$r = r_{MM_0} = \sqrt{(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2} , \qquad (7\ 2.12)$$

$$r_1 = r_{MM_1} = \sqrt{(x - x_0)^2 + (y - y_0)^2 + (z + z_0)^2} ,$$

where $M_1(x_0, y_0, -z_0)$ is the image of the point $M_0(x_0, y_0, z)$ in the plane $z = 0$.

At the moment we shall not go into the question of the applicability of the above formulas for an unbounded region. We can easily establish it in the case $c < 0$. In the case $c > 0$, however, the problem for an unbounded space is related to the "radiation principle." Therefore, we shall concern ourselves with it in the following paragraphs.

For Green's function $G(M, M_0)$ which is defined for an arbitrary region T, the reciprocity principle expressed by the equation

$$G(M, M_0) = G(M_0, M)$$

holds. The proof is a verbatim repetition of the corresponding proofs for the case of the Laplace differential equation (Section 4-2).

For two independent variables the equation for the function $v_0(r)$ has the form

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{dv_0}{dr}\right) + k^2 v_0 = 0$$

and

$$\frac{d^2 v_0}{dr^2} + \frac{1}{r}\frac{dv_0}{dr} + k^2 v_0 = 0 ,$$

that is, the form of the Bessel equation of *zero*th order, whose general solution can be written as follows (see Appendix):

$$v_0(r) = C_1 H_0^{(1)}(kr) + C_2 H_0^{(2)}(kr) .$$

Here $H_0^{(1)}(kr)$ and $H_0^{(2)}(kr)$ are Hankel functions of the *zero*th order of first and second kind.

The functions $H_0^{(1)}(kr)$ and $H_0^{(2)}(kr)$ for $r = 0$ have a logarithmic singularity:

$$H_0^{(1)}(\rho) = -\frac{2i}{\pi} \ln \frac{1}{\rho} + \cdots ,$$

$$H_0^{(2)}(\rho) = \frac{2i}{\pi} \ln \frac{1}{\rho} + \cdots , \quad (\rho = kr) ;$$

[22] In the case of the sphere the method of electrostatic images for $c \neq 0$ is not applicable.

the dots denote the members which remain finite for $\rho = 0$. At infinity (for $\rho \to \infty$) the behavior of the Hankel functions is characterized by the asymptotic representations

$$H_0^{(1)}(\rho) = \sqrt{\frac{2}{\pi\rho}} e^{i(\rho - \pi/4)} + \cdots,$$

$$H_0^{(2)}(\rho) = \sqrt{\frac{2}{\pi\rho}} e^{-i(\rho - \pi/4)} + \cdots;$$

the dots here denote those members which are smaller than $1/\rho$ and of higher order.

Consequently, the equation $\Delta v + k^2 v = 0$ possesses two fundamental solutions

$$v_0(r) = \begin{matrix} H_0^{(1)}(kr), \\ H_0^{(2)}(kr), \end{matrix}$$

which have a logarithmic singularity and correspond to the functions e^{ikr}/r and e^{-ikr}/r in three-dimensional space.

The choice of the fundamental solutions depends on the form of the radiation condition at infinity (see Section 7-3.4). If a time dependence in the form $e^{i\omega t}$ is assumed, then the function $H_0^{(2)}(kr)$ defines a divergent cylindrical wave. If we assume a time dependence $e^{-i\omega t}$, on the other hand, the divergent wave is defined by the function $H_0^{(1)}(kr)$.

If $c = -\kappa^2 < 0$, then the linearly independent solutions of the equation

$$\frac{d^2 v_0}{dr^2} + \frac{1}{r}\frac{dv_0}{dr} - \kappa^2 v_0 = 0$$

are the cylindrical functions with complex arguments

$$I_0(\kappa r) \quad \text{and} \quad K_0(\kappa r).$$

The function $I_0(kr)$ for $r = 0$ is bounded and for $r \to \infty$ increases exponentially; the function $K_0(\kappa r)$ at the point $r = 0$ has the logarithmic singularity

$$K_0(\rho) = \ln \frac{1}{\rho} + \cdots$$

and is therefore the fundamental solution sought. At infinity it behaves according to

$$K_0(\rho) = \sqrt{\frac{\pi}{2\rho}} e^{-\rho} + \cdots.$$

We shall not continue our discussion of Green's formula and the concept of the Green's function G for the case of two independent variables, as we would merely be repeating previous discussions.

3. Potentials

In Chapter 4, we considered the potential for the equation $\Delta u = 0$. Such potentials can also be given for the equation $\Delta u - \kappa^2 u = 0$.

The volume potential (for the equation $\Delta u - \kappa^2 u = 0$) is denoted by an integral of the form

$$V(M) = \int_T \rho(P) \frac{e^{-\kappa r}}{r} d\tau_P ,$$
$$r = r_{MP} = \sqrt{(x-\xi)^2 + (y-\eta)^2 + (z-\zeta)^2} ; \quad d\tau_P = d\xi\, d\eta\, d\zeta , \tag{7-2.13}$$

where $\rho(P)$ is the density of the potential.

We now formulate the principal properties of volume potentials. The proof of these properties is analogous to the proofs in Chapter 4.

1. Exterior to the region T, the function $V(M)$ satisfies the equation

$$\Delta V - \kappa^2 V = 0 .$$

2. Interior to the region T, the integral (7-2.13) converges. Moreover, the integral which is obtained by formal differentiation of $V(M)$ under the integral sign also converges:

$$\int_T \rho(\xi, \eta, \zeta) \frac{\partial}{\partial x}\left[\frac{e^{-\kappa r}}{r}\right] d\xi\, d\eta\, d\zeta , \quad \text{etc.}$$

3. The function $V(x, y, z)$ is differentiable and its derivatives can be obtained by differentiation under the integral sign:

$$\frac{\partial V}{\partial x} = \int_T \rho(\xi, \eta, \zeta) \frac{\partial}{\partial x}\left[\frac{e^{-\kappa r}}{r}\right] d\xi\, d\eta\, d\zeta , \quad \text{etc.}$$

For the proof of the differentiability of $V(x, y, z)$ it is sufficient to assume the boundedness of the function $\rho(M)$. From this condition follows, in particular, the differentiability of $V(M)$ at the points of the boundary Σ of T, in which $\rho(M)$ is naturally discontinuous.

4. At all interior points of the region T in whose neighborhood the density $\rho(M)$ is differentiable, the potential $V(M)$ has second derivatives and the equation

$$\Delta V - \kappa^2 V = -4\pi\rho(M)$$

holds.

5. Under the assumption of the uniform boundedness of $\rho(M)$, the first derivatives of the spatial potential are represented by uniformly convergent integrals. Thus the first derivatives in the entire space, including the points of the boundary Σ, are continuous functions.

With the help of the spatial potentials the solution of the boundary-value problem for the inhomogeneous equation $\Delta u - \kappa^2 u = -f$ can be represented in the form

$$u(M) = V(M) + u_1(M) ,$$

where $V(M)$ is the spatial potential with the density $\rho = f/(4\pi)$, and $u_1(M)$ is the solution of the boundary-value problem for the homogeneous equation $\Delta u_1 - \kappa^2 u_1 = 0$.

We shall now go briefly into the properties of the potential of a single and double layer. The potential of a double layer is defined by the integral

$$W(M) = \int_{\Sigma} \mu(P) \frac{\partial}{\partial \nu_P} \left[\frac{e^{-\kappa r}}{r} \right] d\sigma_P , \quad (r = r_M P) , \tag{7-2.14}$$

in which $\mu(P)$ is the surface density of the potential $W(M)$.

In regard to the properties of the potential of a double layer, we recall those enumerated in Section 4-5:

1. Exterior to the surface Σ, the potential of a double layer everywhere satisfies the homogeneous equation $\Delta W - \kappa^2 W = 0$.

2. The potential of a double layer converges at the points of the boundary if Σ belongs to the class of Liapunoff surfaces.

3. The function $W(M)$ is discontinuous at points of the surface Σ; there, the following relations hold:

$$W_I(M_0) = W(M_0) + 2\pi\mu(M_0) ,$$
$$W_A(M_0) = W(M_0) - 2\pi\mu(M_0) .$$

Here $W_I(M_0)$ denotes the limit value of $W(M)$ when M approaches M_0 from the interior of the region T, and $W_A(M_0)$ is the limit value of $W(M)$ when M approaches M_0 from the exterior.

The potential of a single layer defined as a surface integral

$$V(M) = \int_{\Sigma} e(P) \frac{e^{-\kappa r}}{r} d\sigma_P , \quad (r = r_{MP}) \tag{7-2.15}$$

possesses the following properties:

1. Exterior to the surface Σ, the potential of a single layer everywhere satisfies the homogeneous equation $\Delta V - \kappa^2 V = 0$.

2. The integral converges uniformly on Σ and defines a function $V(M)$, which is continuous in the entire space.

3. The normal derivative of the potential of a single layer in the case of a Liapunoff surface satisfies the relations [see formulas (4-5.48)]

$$\left(\frac{\partial V}{\partial \nu} \right)_I = V_0 + 2\pi\rho(M_0) ,$$
$$\left(\frac{\partial V}{\partial \nu} \right)_A = U_0 - 2\pi\rho(M_0) .$$

Here

$$\left(\frac{\partial V}{\partial \nu} \right)_I \quad \text{and} \quad \left(\frac{\partial V}{\partial \nu} \right)_A$$

denote the limit values of the normal derivative by approaching a point M_0 of the surface Σ from the interior (respectively, exterior to the surface Σ; ν is the exterior normal), and

$$U_0(M_0) = \int_{\Sigma} \rho(P) \frac{\partial}{\partial \nu} \left[\frac{e^{-\kappa r}}{r} \right] d\sigma_P , \quad (r = r_{M_0 P}) .$$

With the help of the surface potentials it is possible for a reasonably extensive class of surfaces (for example, for Liapunoff surfaces) to cast boundary-value problems in terms of integral equations. Here we consider the first interior boundary-value problem for the equation $\Delta u - \kappa^2 u = 0$ with the boundary condition $u|_{\Sigma} = f$, and we assume that the desired solution can be represented as a potential of a double layer:

$$u(M) = W(M) = \int_\Sigma \mu(P) \frac{\partial}{\partial \nu_P} \left[\frac{e^{-\kappa r}}{r} \right] d\sigma_P . \tag{7-2.14}$$

This function, as was explained above, satisfies the homogeneous equation $\Delta u - \kappa^2 u = 0$ in T. If we now require that the boundary condition $u|_\Sigma = f$ be satisfied, then for the determination of the function $\mu(M)$ we arrive at the integral equation

$$2\pi\mu(M) + \int_\Sigma \mu(P) \frac{\partial}{\partial \nu_P} \left[\frac{e^{-\kappa r}}{r} \right] d\sigma_P = f(M) .$$

This equation represents a linear Fredholm integral equation of the second kind. We shall not go into the question of the existence and the uniqueness of the solution of this integral equation.

As with the Laplace differential equation, the difference method is applicable for the equation $\Delta u - \kappa^2 u = 0$.

7-3. PROBLEMS FOR AN UNBOUNDED REGION: RADIATION PRINCIPLE

1. The equation $\Delta v + cv = -f$ in an unbounded space

Now we turn to the solution of the equation

$$\Delta v + cv = -f \tag{7-3.1}$$

in an unbounded space. For simplicity we assume f to be different from 0 only in a fixed bounded region. The character of the solution of this equation depends essentially on the sign of the coefficient c.

Next we consider the case $c = \kappa^2 < 0$. The solution of the equation

$$\Delta v - \kappa^2 v = -f$$

can be represented in the form of spatial potentials

$$v_1(M) = \int_T f(P) \frac{e^{-\kappa r}}{4\pi r} d\tau_P \quad \text{and} \quad v_2(M) = \int_T f(P) \frac{e^{\kappa r}}{4\pi r} d\tau_P .$$

Therefore the solution of Eq. (7-3.1) without auxiliary conditions at infinity cannot be uniquely determined. By analogy with exterior problems for the Laplace differential equation, we seek those solutions of Eq. (7-3.1) which are equal to zero at infinity. This requirement is sufficient for $v_1(M)$, but not for $v_2(M)$.

We prove the following uniqueness theorem:

Theorem. The equation

$$\Delta v - \kappa^2 v = -f$$

can possess at the most one solution which is equal to zero at infinity.[23]

[23] By the phrase "a function is equal to zero at infinity" we understand the following: For each $\varepsilon > 0$, there exists an $r(\varepsilon)$ such that for an arbitrary point $M(r, \theta, \varphi)$ with $r > r(\varepsilon)$ the inequality $|u(M)| < \varepsilon$ always holds, that is, $u(M)$ for $r \to \infty$ approaches zero uniformly.

If we assume that there exist two different solutions $\bar{v}(M)$ and $\bar{\bar{v}}(M)$, then we can consider their difference $w = \bar{v} - \bar{\bar{v}}$. By assumption there exists a point M_0 such that $w(M_0) = A \neq 0$.

We set $A > 0$. Because $w(M) \to 0$ for $M \to \infty$, an R_0 can be found such that $w < A/2$ holds for $r > R_0$. From this it follows that the point M_0 lies within a sphere T_{R_0} with radius R_0 and that the function $w(M)$ assumes its maximum within T_{R_0}. However, this contradicts the principle of the maximum which, as is well known, is valid for our equation (See Section 7-2.1). The uniqueness theorem is thus proved. We proceed analogously in the case $A < 0$.

Now we consider the case $c = k^2 > 0$. The functions

$$v_1(M) = \int_T f(P)\frac{e^{-ikr}}{4\pi r}\,d\tau_P \quad \text{and} \quad v_2(M) = \int_T f(P)\frac{e^{ikr}}{4\pi r}\,d\tau_P$$

are again solutions of Eq. (7-3.1). In this case, however, both functions vanish at infinity. Therefore it is necessary to introduce additional conditions at infinity by which the solution of Eq. (7-3.1) may be uniquely determined. For the rest of this chapter we shall concern ourselves with these conditions.

2. The principle of limiting absorption†

Problems of forced vibrations with decreasing amplitude lead to the equation

$$\Delta u = \frac{1}{a^2}u_{tt} + \beta u_t - F(M, t), \quad (\beta > 0). \tag{7-3.2}$$

We shall assume $F(M, t)$ to be a time periodic function; that is, let $F(M, t) = f(M)e^{i\omega t}$. In this case Eq. (7-3.2) has periodic solutions of the form

$$u(M, t) = v(M)e^{i\omega t}. \tag{7-3.3}$$

The function $v(M)$ must, obviously, satisfy the equation

$$\Delta v + cv = -f(M). \tag{7-3.4}$$

Here $c = k^2 - i\beta\omega$ is a complex quantity.

Equation (7-3.4) together with the complex valued coefficient c is denoted as an equation with complex absorption of the first kind when the imaginary part of c is smaller than zero ($Imc < 0$), and it is regarded as an equation with complex absorption of the second kind when it is larger than zero ($Imc > 0$). The first kind corresponds to the time dependence $e^{i\omega t}$, the second kind to the $e^{-i\omega t}$.

The fundamental solutions of this equation depend now only on r and have the form

$$\bar{v}_0(r) = \frac{e^{-iqr}}{r} \quad \text{and} \quad \bar{\bar{v}}_0(r) = \frac{e^{iqr}}{r},$$

† Editor's note: A very general treatment of this principle is given by D. M. Eidus. See New York Univ. Research Report No. EM-191, Courant Institute of Mathematical Sciences, Div. of EM Research, Aug. 1963 (Trans. by C. L. Hill).

where

$$q = \sqrt{\left(\frac{\omega}{a}\right)^2 - i\beta\omega}$$
$$= \sqrt{\frac{\sqrt{k^4 + \beta^2\omega^2} + k^2}{2}} - i\sqrt{\frac{\sqrt{k^4 + \beta^2\omega^2} - k^2}{2}} = q_0 - iq_1 . \tag{7-3.5}$$

The sign of the root is chosen so that $q_1 > 0$. Consequently, we have

$$\bar{v}_0(r) = \frac{e^{-iq_0 r}}{r}e^{-q_1 r}, \quad \bar{\bar{v}}_0(r) = \frac{e^{iq_0 r}}{r}e^{q_1 r} .$$

At infinity, of course, only the function $\bar{v}_0(r)$ is bounded; the function $\bar{\bar{v}}_0(r)$ increases without bound for $r \to \infty$, so that it has no direct physical significance.

The volume potential

$$\bar{v}(M) = \int_T f(P)\frac{e^{-iq_0 r}}{4\pi r}e^{-q_1 r}\, d\tau_P , \quad (r = r_{MP}) \tag{7-3.6}$$

represents the single solution of Eq. (7-3.4) which is equal to zero at infinity. The limiting value of $\bar{v}(M)$ for $\beta \to 0$ is equal to

$$v(M) = \lim_{\beta \to 0} \bar{v}(M) = \int_T f(P)\frac{e^{-ikr}}{4\pi r}\, d\tau_P , \quad (r = r_{MP}) ,$$

since $q_0 \to k$ and $q_1 \to 0$ as $\beta \to 0$. For the chosen time dependence $e^{i\omega t}$, $q_0 > 0$, since the sign of q_0 is related to the sign of q_1 by means of the relation $2q_0q_1 = \beta\omega$ (we have $q^2 = c$).

If we take the dependence of the time factor in the form $e^{-i\omega t}$ ($Imc > 0$), then a negative q_0 corresponds to the positive value of q_1 and the limit value of q_0 as $\beta \to 0$ is equal to $-k$.

To characterize a solution of

$$\Delta v + k^2 v = -f$$

we therefore must require, in addition, that $v(M)$ be a limit value of a bounded solution of the vibration equation with complex absorption of the first kind, when the imaginary part of the absorption coefficient approaches zero.[24]

3. The principle of limiting amplitude†

For the investigation of forced vibrations induced by periodic forces, we often consider, in particular, the vibration equation

$$\Delta v + k^2 v = -f \tag{7-3.7}$$

(see Section 7-1.1).

We consider the wave equation whose right-hand side is periodic

$$\Delta u - \frac{1}{a^2}u_{tt} = -F , \quad (F = fe^{i\omega t}) . \tag{7-3.8}$$

[24] See A. G. Sveshnikov, "Radiation Principles," *Doklady* 73, 5, 1950.

† Editor's note: See D. M. Eidus, *loc. cit.*, the treatment is based on the methods of functional analysis (see also references contained therein).

In order to determine the solution we must still add initial conditions to the equations, for example, the homogeneous conditions

$$u(M, 0) = 0\,, \quad u_t(M, 0) = 0\,. \tag{7-3.9}$$

The function $u(M, t)$ in the initial stage of the process would not be rigorously periodic. However, in the course of time, the system carries out periodic vibrations with the frequency of the acting force, that is, the solution $u(M, t)$ takes the form

$$u(M, t) = v(M)e^{i\omega t}\,. \tag{7-3.10}$$

Here $v(M)$ represents the limit amplitude of the vibration, that is,

$$v(M) = \lim_{t\to\infty} ue^{-i\omega t}\,,$$

and $v(M)$ satisfies the vibration equation

$$\Delta v + k^2 v = -f\,, \quad \left(k = \frac{\omega}{a}\right).$$

The requirement that $v(M)$ be the limit amplitude of vibration for homogeneous initial conditions therefore represents an additional condition which guarantees the unique solvability of the vibration equation.

Thus we arrive at the following problem:

Determine the solution of the vibration equation $\Delta v + k^2 v = -f$ which is the limit amplitude for the solution of the wave equation

$$\Delta u - \frac{1}{a^2} u_{tt} = -f(M)e^{i\omega t} \tag{7-3.8'}$$

with the initial conditions

$$u(M, 0) = 0\,, \quad u_t(M, 0) = 0\,. \tag{7-3.9}$$

We shall give an explicit representation for the limit amplitude. For this purpose we first determine the solution of the wave equation (7-3.8′) with homogeneous initial conditions by using the formula obtained in Chapter 5 (see Section 5-2.6):

$$u(M, t) = \frac{1}{4\pi}\int_{T_{at}^M} \frac{f(P)e^{i\omega(t-r/a)}}{r}\, d\tau_P\,, \quad (r = r_{MP})\,.$$

Here T_{at}^M is a sphere of radius at with center at M.

Now let $f(P)$ be different from zero only within a fixed bounded region T_0. Then, for the limiting amplitude we obtain the expression

$$\begin{aligned} v(M) = \lim_{t\to\infty} u(M, t)e^{-i\omega t} &= \lim_{t\to\infty} \frac{1}{4\pi}\int_{T_{at}^M} \frac{e^{-ikr}}{r} f(P)\, d\tau_P \\ &= \frac{1}{4\pi}\int_{T_0} f(P)\frac{e^{-ikr}}{r}\, d\tau_P\,, \quad (r = r_{MP})\,. \end{aligned}$$

Thus the limit amplitude represents a spatial potential which is determined by the fundamental solution e^{-ikr}/r. This fundamental solution corresponds to the diverging wave $e^{i(\omega t - kr)}/r$.

The principle of limiting amplitude thus leads to the same result mathematically as the principle of limiting absorption. This is quite understandable since both principles denote that fundamental solution which corresponds to a diverging wave.

4. Radiation conditions

The preceding section stated the general physical considerations which permitted us to find a divergent wave corresponding to the solution of the vibration equation. The method used, however, required the solution of auxiliary problems. We shall now establish an analytical condition by which a divergent wave is characterized.

The plane wave propagating in the direction of the x-axis has the form

$\bar{u} = f\left(t - \frac{x}{a}\right)$ direct wave (propagating in the direction of the positive x-axis);

$\bar{u} = f\left(t + \frac{x}{a}\right)$ reverse wave (propagating in the direction of the negative x-axis).

A direct wave is characterized by the expression

$$\frac{\partial \bar{u}}{\partial x} + \frac{1}{a}\frac{\partial \bar{u}}{\partial t} = 0 ,$$

and the reverse wave by the expression[25]

$$\frac{\partial \bar{u}}{\partial x} - \frac{1}{a}\frac{\partial \bar{u}}{\partial t} = 0 .$$

For a forced vibration,

$$u = v(x)e^{i\omega t} .$$

These expressions assume the form

$$\frac{\partial \bar{v}}{\partial x} + ik\bar{v} = 0 , \qquad \left(k = \frac{\omega}{a}\right) \qquad \text{for the direct wave ,} \tag{7-3.11}$$

$$\frac{\partial \bar{\bar{v}}}{\partial x} - ik\bar{\bar{v}} = 0 , \qquad \text{for the reverse wave .} \tag{7-3.12}$$

We turn now to a consideration of spherical waves. If a spherical wave is generated by sources which are distributed in a bounded part of space, then the spherical wave at a sufficiently large distance from the source region approaches a plane wave whose amplitude decreases as $1/r$. We can further assume that a divergent spherical wave satisfies the expression[26]

$$\frac{\partial u}{\partial r} + \frac{1}{a}\frac{\partial u}{\partial t} = o\left(\frac{1}{r}\right) . \tag{7-3.13}$$

[25] These expressions are partial differential equations of the first order whose solutions have the form of a direct or reverse wave.

[26] In the following we use the notations: $0(\xi)$ is a quantity which approaches 0 to the same order as ξ; $o(\xi)$ is a quantity which for $\xi \to 0$ approaches 0 with a higher order than ξ (Landau symbol).

The corresponding condition for a converging spherical wave reads

$$\frac{\partial u}{\partial r} - \frac{1}{a}\frac{\partial u}{\partial t} = o\left(\frac{1}{r}\right). \tag{7-3.14}$$

For the amplitude of a forced vibration these conditions yield

$$\frac{\partial v}{\partial r} + ikv = o\left(\frac{1}{r}\right) \quad \text{for the diverging spherical wave,} \tag{7-3.15}$$

$$\frac{\partial v}{\partial r} - ikv = o\left(\frac{1}{r}\right) \quad \text{for the converging spherical wave.} \tag{7-3.16}$$

Expressions (7-3.15) and (7-3.16) result from the assumption that, at sufficiently large distances, each sperical wave is similar to a plane wave whose amplitude decreases as $1/r$. We shall now investigate the validity of this assumption.

Case 1. For the case of a point source placed at the origin of coordinates, the validity of our assumption can easily be recognized. The wave, indeed, has the form

$$u(r, t) = \frac{e^{i(\omega t - kr)}}{r} = v_0(r)e^{i\omega t},$$

so that

$$\frac{\partial v_0}{\partial r} + ikv_0 = o\left(\frac{1}{r}\right)$$

holds.

Case 2. The spherical wave can be generated by a point source which is found at a point M_0. The amplitude of the spherical wave is equal to

$$v_0(M) = \frac{e^{-ikr_0}}{r_0},$$

where

$$r_0 = \sqrt{r^2 + \rho_0^2 - 2r\rho_0 \cos\vartheta}$$

is the distance between the points M and M_0 (Figure 88). Furthermore

$$\frac{\partial r_0}{\partial r} = \frac{r - \rho_0 \cos\vartheta}{r_0} \approx 1 + O\left(\frac{1}{r}\right).$$

According to Case 1 we have

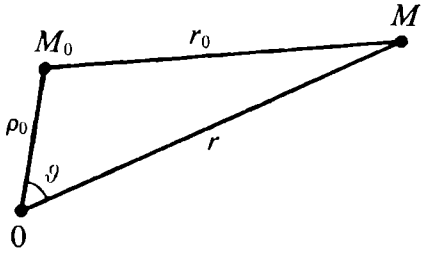


FIG. 88

$$\frac{\partial v_0}{\partial r_0} + ikv_0 = o\left(\frac{1}{r_0}\right).$$

Now we prove the validity of formula (7-3.15):

$$\boldsymbol{L}(v_0) = \frac{\partial v_0}{\partial r} + ikv_0 = o\left(\frac{1}{r}\right). \tag{7-3.15'}$$

We have

$$\frac{\partial v_0}{\partial r} = \frac{\partial v_0}{\partial r_0}\frac{\partial v_0}{\partial r} = \frac{\partial v_0}{\partial r_0}\left(1 + O\left(\frac{1}{r}\right)\right) = \frac{\partial v_0}{\partial r_0} + o\left(\frac{1}{r}\right),$$

because

$$\frac{\partial v_0}{\partial r_0} O\left(\frac{1}{r}\right) = o\left(\frac{1}{r}\right).$$

From this and from Case 1, we find

$$\boldsymbol{L}(v_0) = \frac{\partial v_0}{\partial r_0} + ikv_0 + o\left(\frac{1}{r}\right) = o\left(\frac{1}{r}\right),$$

which was to be proved.

Case 3. We now prove that the volume potential

$$v(M) = \int_T f(P)\frac{e^{-ikr}}{4\pi r}\,d\tau_P, \quad (r = r_{MP}) \tag{7-3.16'}$$

satisfies the condition (7-3.15). Obviously,

$$\boldsymbol{L}(v) = \int_T f(P)\boldsymbol{L}\left(\frac{e^{-ikr}}{4\pi r}\right)d\tau_P = \int_T f(P)o\left(\frac{1}{r}\right)d\tau_P = o\left(\frac{1}{r}\right).$$

Formula (7-3.10) represents the amplitude of a diverging wave which is generated by sources which are distributed arbitrarily in the interior of a fixed bounded region T. We saw that the function $v(M)$ satisfies the vibration equation

$$\Delta v + k^2 v = -f(M),$$

and for $r \to \infty$, approaches zero as $1/r$. Moreover, as was shown, it satisfies the necessary auxiliary condition

$$\frac{\partial v}{\partial r} + ikv = o\left(\frac{1}{r}\right)$$

at infinity.

We shall now prove the following statement:

There exists exactly one solution of the vibration equation

$$\Delta v + k^2 v = -f(M)$$

[$f(M)$ is different from zero only within a fixed bounded spatial region] which at infinity satisfies[27] the conditions

[27] I. N. Vekua showed that the first of these two conditions is a consequence of the second; see I. N. Vekua, *Works of the Mathematical Institute in Tiflis*, Vol. 12, 1943.

$$v = O\left(\frac{1}{r}\right), \quad \frac{\partial v}{\partial r} + ikv = o\left(\frac{1}{r}\right). \tag{7-3.17}$$

Let us assume that two different solutions v_1 and v_2 exist. Their difference

$$w = v_1 - v_2$$

then satisfies the homogeneous vibration equation and the conditions (7-3.17). Let Σ_R be a sphere of radius R. In the following discussion we shall let R become unbounded. By applying Green's formula to the functions $w(M)$ and $v_0(M) = e^{-ikr}/(4\pi r)$, for a point M_0 which lies in the interior of Σ, there results the expression

$$w(M_0) = \int_{\Sigma_R} \left(v_0 \frac{\partial w}{\partial r} - w \frac{\partial v_0}{\partial r} \right) d\sigma .$$

The conditions (7-3.17) for $v_0(r)$ and $w(M)$ give

$$\begin{aligned} v_0 \frac{\partial w}{\partial r} - w \frac{\partial v_0}{\partial r} &= v_0 \left[-ikw + o\left(\frac{1}{r}\right) \right] - w \left[-ikv_0 + o\left(\frac{1}{r}\right) \right] \\ &= v_0 o\left(\frac{1}{r}\right) - wo\left(\frac{1}{r}\right) = o\left(\frac{1}{r_2}\right). \end{aligned}$$

Thus we find

$$w(M_0) = \int_{\Sigma_R} o\left(\frac{1}{r_2}\right) d\sigma \to 0 \quad \text{for} \quad R \to \infty .$$

Since M_0 is arbitrarily selected, the unique solvability of our problem follows.

The conditions

$$v = O\left(\frac{1}{r}\right), \quad \frac{\partial v}{\partial r} + ikv = o\left(\frac{1}{r}\right) \tag{7-3.17}$$

are generally called the radiation conditions or the Sommerfeld conditions.

Let us note that the radiation conditions for unbounded regions which do not coincide with the entire space can assume forms which differ from the Sommerfeld conditions.

Conditions (7-3.17), therefore, represent the analytic form of radiation conditions for an unbounded space which require no physical principle for their support. Thus these conditions can be formulated for regions with complicated forms.

The radiation conditions which result from the introduction of an infinitesimally small complex absorption in the vibration equation were first used by W. S. Ignatowsky.[28]

The principle of infinitesimally small complex absorption can be applied to unbounded regions of different forms and also to complicated problems.

For plane problems which correspond to the equation

$$\Delta_2 v + k^2 v = 0 , \tag{7-3.18}$$

[28] W. S. Ignatowsky, *Ann. Phys.*, 18, 1905.

the radiation conditions at infinity assume the form

$$v = O\left(\frac{1}{\sqrt{r}}\right), \quad \lim_{r\to\infty} \sqrt{r}\left(\frac{\partial v}{\partial r} + ikv\right) = 0 . \tag{7-3.19}$$

The simpliest solutions of this equation are the Hankel functions of *zero*th order

$$H_0^{(1)}(kr) \quad \text{and} \quad H_0^{(2)}(kr)$$

(see Section A-2.3). From the asymptotic representations

$$H_\nu^{(1)}(kr) = \sqrt{\frac{2}{\pi kr}} e^{i[kr-(\pi/2)\nu-(\pi/4)]}\left[1 + O\left(\frac{1}{r}\right)\right],$$

$$H_\nu^{(2)}(kr) = \sqrt{\frac{2}{\pi kr}} e^{-i[kr-(\pi/2)\nu-(\pi/4)]}\left[1 + O\left(\frac{1}{r}\right)\right]$$

and the recursion relations

$$\frac{dH_0^{(1)}}{dx} = -H_1^{(1)}(x), \quad \frac{dH_0^{(2)}}{dx} = -H_1^{(2)}(x)$$

we see that only the function $H_0^{(2)}(kr)$ satisfies the radiation condition.

The function $H_0^{(2)}(kr)$ therefore satisfies Eq. (7-3.18) and the radiation condition (7-3.19). Moreover, $H_0^{(2)}(kr)$ for $r \to 0$ has a logarithmic singularity. As we have already mentioned in Section 7-2, $H_0^{(2)}(kr)$ in the case of two independent variables, plays the role of a Green's function for the vibration equation (7-3.18). The solution of the inhomogeneous equtaion

$$\Delta_2 v + k^2 v = -f$$

can therefore be expressed by

$$v(M) = -\frac{i}{4}\int_S\int H_0^{(2)}(kr_{MP}) f(P)\, d\sigma_P .$$

Here S is the region in which the function $f(M)$ is different from zero.

7-4. PROBLEMS OF THE MATHEMATICAL THEORY OF DIFFRACTION

1. Statement of the problem

The propagation of wave processes (electromagnetic, elastic, accoustics, etc.) is accompanied by an entire series of typical phenomena (diffraction, refraction, reflection, etc.) The solution of these problems either leads directly to the solution of the vibration equation in an inhomogeneous medium or has much in common with the solution of the equation

$$\frac{\partial}{\partial x}\left(p\frac{\partial v}{\partial x}\right) + \frac{\partial}{\partial y}\left(p\frac{\partial v}{\partial y}\right) + \frac{\partial}{\partial z}\left(p\frac{\partial v}{\partial z}\right) + \rho\omega^2 v = -f, \quad (p > 0) . \tag{7-4.1}$$

in which p and ρ are parameters dependent on the properties of the media.

The greatest interest from a physical standpoint is shown in the case of piecewise constant parameters p and ρ. The corresponding mathematical problems are the following:

In an unbounded space, consider a sequence of bounded regions T_i with constant parameter values p_i and ρ_i. That part of T_0 in the space which lies outside the region T_i is likewise homogeneous ($p_0 = \text{const}$, $\rho_0 = \text{const}$). Then, in the interior of each subregion T_i the vibration has the usual form

$$\Delta v_i + k_i^2 v_i = -f_i \text{ in } T_i\,, \quad (i = 0, 1, \cdots, n)\,, \tag{7-4.2}$$

where u_i is the value of the sought function u in the interior of T_i. Furthermore

$$k_i^2 = \frac{\rho_i \omega^2}{p_i}\,, \quad f_i = \frac{\bar{f}}{p_i}$$

in T_i. On the bounding surfaces Σ_i of T, the differential equations are replaced by the conditions[29]

$$\begin{aligned} v_i &= v_0 \text{ on } \Sigma_i\,, \\ p_i \frac{\partial v_i}{\partial n} &= p_0 \frac{\partial v_0}{\partial n} \text{ on } \Sigma_i \quad (i = 1, 2, \cdots, n)\,. \end{aligned} \tag{7-4.3}$$

At infinity the function v_0, as the solution of the vibration equation $\Delta v + k_0^2 v = -f_0$ in T_0, satisfies the radiation conditions

$$v_0(M) = O\left(\frac{1}{r}\right)\,, \quad \frac{\partial v_0}{\partial r} + ikv_0 = o\left(\frac{1}{r}\right)\,. \tag{7-4.4}$$

Later we shall show that conditions (7-4.3) together with the radiation conditions (7-4.4) are adequate for a unique definition of the function v in the entire space. The problem formulated in this section represents the simpliest problem in the mathematical theory of diffraction.

2. Uniqueness of the solutions

It will be shown in the following discussion that the problem of the mathematical theory of diffraction, formulated in Section 7-4.1, possesses a uniquely determined solution. To simplify the proof we assume that the homogeneity of the medium is perturbed only by a single body T_1. This body will be bounded by the closed surface Σ_1, and the region T_0 can lie outside this surface. Therefore, we do not assume that T_1 is a simply connected region. Thus we have

Lemma. There exists at most one function with the following properties:

1. It satisfies the equations

$$\begin{aligned} L_0(v_0) &= \Delta v_0 + k_0^2 v_0 = -f_0 \text{ in } T_0\,, \\ L_1(v_1) &= \Delta v_1 + k_1^2 v_1 = -f_1 \text{ in } T_1\,. \end{aligned} \tag{7-4.2'}$$

2. It satisfies the conditions

$$v_1 = v_0\,, \quad p_1 \frac{\partial v_1}{\partial \nu} = p_0 \frac{\partial v_0}{\partial \nu} \tag{7-4.3}$$

[29] Here, for simplicity, we assume that the subregion T_i has only one boundary in common with the surrounding medium.

on the boundary Σ.

3. At infinity the radiation conditions

$$v_0 = O\left(\frac{1}{r}\right), \quad \frac{\partial v_0}{\partial r} + ikv_0 = o\left(\frac{1}{r}\right) \tag{7-4.4}$$

are satisfied.

Proof. We assume that two different solutions

$$\bar{v} = \{\bar{v}_1, \bar{v}_0\} \quad \text{and} \quad \dot{\bar{v}} = \{\dot{\bar{v}}_1, v_0\}$$

exist and consider their difference

$$w = \{w_1, w_0\},$$

where

$$w_1 = \bar{v}_1 - \dot{\bar{v}}_1, \quad w_0 = \bar{v}_0 - v_0.$$

The difference then satisfies the homogeneous equation and the auxiliary conditions

$$\boldsymbol{L}_0(w_0) = 0 \text{ in } T_0, \quad \boldsymbol{L}_1(w_1) = 0 \text{ in } T_1, \tag{7-4.2'}$$

$$w_1 = w_0, \quad p_1\frac{\partial w_1}{\partial \nu} = p_0\frac{\partial w_0}{\partial \nu} \text{ on } \Sigma_1, \tag{7-4.3'}$$

$$w_0 = O\left(\frac{1}{r}\right), \quad \frac{\partial w_0}{\partial r} + ikw_0 = o\left(\frac{1}{r}\right) \text{ for } r \to \infty. \tag{7-4.4'}$$

Let w_0^* and w_1^* be complex functions conjugate to w_0 and w_1, respectively. Obviously these satisfy the homogeneous equations (7-4.2'), the conditions (7-4.3'), as well as the radiation conditions

$$w_0^* = O\left(\frac{1}{r}\right), \quad \frac{\partial w_0^*}{\partial r} - ikw_0^* = o\left(\frac{1}{r}\right). \tag{7-4.4''}$$

Now let Σ_R be a sphere of sufficiently large radius which contains the region T_1. Let T_R be the region bounded by the surfaces Σ_1 and Σ_R.

By application of Green's formula to the functions w_1, w_1^* in the region T_1, and to w_0, w_0^* in the region T_R, we obtain

$$\int_{T_1} (w_1 \boldsymbol{L}_1(w_1^*) - w_1^* \boldsymbol{L}_1(w_1))\, d\tau = \int_{\Sigma_1} \left(w_1\frac{\partial w_1^*}{\partial \nu_1} - w_1^*\frac{\partial w_1}{\partial \nu_1}\right) d\sigma,$$

$$\int_{T_R} (w_0 \boldsymbol{L}_0(w_0^*) - w_0^* \boldsymbol{L}_0(w))\, d\tau$$

$$= \int_{\Sigma_1} \left(w_0\frac{\partial w_0^*}{\partial \nu_0} - w_0^*\frac{\partial w_0}{\partial \nu_0}\right) d\sigma + \int_{\Sigma_R} \left(w_0\frac{\partial w_0^*}{\partial \nu} - w_0^*\frac{\partial w_0}{\partial \nu}\right) d\sigma = 0.$$

Here ν_0 is the exterior normal to T_R, and ν_1 is the exterior normal to T_1. Obviously

$$\frac{\partial}{\partial \nu_0} = -\frac{\partial}{\partial \nu_1}$$

on Σ_1.

Now we multiply the first equation by p_1 and the second by p_0. By ad-

dition of the equations so obtained and by use of conditions (7-4.3′), we then find

$$\int_{\Sigma_R}\left(w_0\frac{\partial w_0^*}{\partial r} - w_0^*\frac{\partial w_0}{\partial r}\right)d\sigma = 0\,.$$

From the radiation conditions we conclude that

$$\frac{\partial w_0^*}{\partial r} = ikw_0^* + o\left(\frac{1}{r}\right), \quad \frac{\partial w_0}{\partial r} = -ikw_0 + o\left(\frac{1}{r}\right).$$

With these expressions for the derivatives, we arrive at the equation

$$2ik\int_{\Sigma_R} w_0 w_0^*\,d\sigma + \int_{\Sigma_R}\left[w_0 o\left(\frac{1}{R}\right) - w_0^* o\left(\frac{1}{R}\right)\right]d\sigma = 0\,.$$

The second integral approaches zero as $R \to \infty$, so that

$$\int_{\Sigma_R} w_0 w_0^*\,d\sigma = \int_{\Sigma_R} |\,Rw_0\,|^2\,d\Omega \to 0,\ R \to \infty\,,\quad (d\Omega = \sin\theta\,d\theta\,d\varphi)\,. \tag{7-4.5}$$

In the Appendix [see Eq. (A-3.93)] is shown that the functions

$$V_m(r, \theta, \varphi) = \zeta_m^{(2)} Y_m(\theta, \varphi)\,,$$

where

$$\zeta_m^{(2)}(\rho) = \sqrt{\frac{2}{\pi\rho}}H^{(2)}_{m+1/2}(\rho)\,,\quad (\rho = k_0 r)$$

and $Y_m(\theta, \varphi)$ denote the spherical functions of the mth degree which satisfy the vibration equation

$$\boldsymbol{L}_0(V_m) = \Delta V_m + k_0^2 V_m = 0$$

and the radiation condition

$$\frac{\partial V_m}{\partial r} + ik_0 V_m = o\left(\frac{1}{r}\right).$$

Now in the region T_R we apply Green's formula to the functions w_0 and V_m and obtain

$$\begin{aligned} 0 &= \int_{T_R}[w_0\boldsymbol{L}(V_m) - V_m\boldsymbol{L}(w_0]\,d\tau \\ &= \int_{\Sigma_1}\left(w_0\frac{\partial V_m}{\partial \nu} - V_m\frac{\partial w_0}{\partial \nu}\right)d\sigma + \int_{\Sigma_R}\left(w_0\frac{\partial V_m}{\partial \nu} - V_m\frac{\partial w_0}{\partial \nu}\right)d\sigma = I_1 + I_R\,. \end{aligned}$$

The second summand I_R, because of the radiation condition, approaches zero as $R \to \infty$ (see the theorem from Section 7-3.4). Since the first integral I_1 is independent of R, it follows that I_1 must be equal to zero. Consequently, $I_R = 0$ for arbitrary R, that is, we have

$$\left.\frac{d\zeta_m^{(2)}(k_0 r)}{dr}\right|_{r=R}\cdot\int_{\Sigma_R} w_0 Y_m(\theta, \varphi)\,d\Omega - \left.\zeta_m^{(2)}\right|_{r=R}\cdot\int_{\Sigma_R}\frac{\partial w_0}{\partial r}Y_m(\theta, \varphi)\,d\Omega = 0\,.$$

With the notation

$$\int_{\Sigma_r} w_0 Y_m(\theta, \varphi)\, d\Omega = \alpha_m(k_0 r)\,,$$

we can write

$$\zeta_m^{(2)\prime}(k_0 R)\alpha_m(k_0 R) - \alpha_m'(k_0 R)\zeta_m^{(2)}(k_0 R) = 0\,.$$

Hence we obtain

$$\alpha_m(k_0 r) = a_m \zeta_m^{(2)}(k_0 r)\,,$$

where a_m is a constant factor.

The completeness relation for the spherical functions

$$\int_{\Sigma_R} |Rw_0|^2\, d\Omega = \sum_{m=0}^{\infty} R^2 \alpha_m^2(k_0 R) \tag{7-4.6}$$

and expression (7-4.5) give

$$R\alpha_m(k_0 R) \to 0 \quad \text{for} \quad R \to \infty\,.$$

On the basis of the asymptotic representation

$$\zeta_m^{(2)}(\rho) \approx \frac{1}{\rho} e^{-i[\rho - (m+1/2)\pi]}\,,$$

the sum of the products $r\zeta_m^{(2)}(k_0 r)$ remains larger than a fixed positive number for large values of r. Consequently, $a_m = 0$, i.e., $\alpha_m(r) \equiv 0$. Because of the completeness relation (7-4.6), it follows that w_0 on the surface of the sphere Σ_{r_0} is equal to zero. Thus, if Σ_{r_0} with radius r_0 surrounds the region T_1, then outside of this sphere $w \equiv 0$. Since the solution of the equation $L = 0$ is analytic,[30] we can conclude that $w_0 \equiv 0$ everywhere in T_1. Furthermore, from conditions (7-4.3) we find that on the surface Σ_1,

$$w_1 = 0 \quad \text{and} \quad \frac{\partial w_1}{\partial \nu} = 0 \tag{7-4.7}$$

holds.

By application of Green's formula to the function w_1 in the region T_1, we know that at an arbitrary point M of the region T_1 we find

$$w_1(M) = \frac{1}{4\pi} \int_{\Sigma_1} \left[\frac{e^{-ik_1 r}}{r} \frac{\partial w_1(P)}{\partial \nu_1} - w_1(P) \frac{\partial}{\partial \nu_1} \left(\frac{e^{-ik_1 r}}{r} \right) \right] d\sigma = 0\,, \tag{7-4.8}$$

where $r = r_{MP}$.

We have therefore proved that in the entire space, $w(M) \equiv 0$, and by this the uniqueness theorem is also proved.

3. Diffraction by a sphere

1. A very important practical class of solutions of the wave equation

$$\Delta u - \frac{1}{a^2} u_{tt} = 0$$

[30] That the function w in the region T_1 is analytic follows from formula (7-2.7′) for complex values $\kappa = ik$ and for a surface Σ which lies entirely in the interior of T_1.

is formed by plane waves. By a plane wave, we mean a solution which depends only on a single spatial coordinate besides the time. The direction of propagation of the wave, therefore, coincides with this coordinate axis. For example, a wave propagating along the x-axis which satisfies the equation

$$u_{xx} - \frac{1}{a^2}u_{tt} = 0 ,$$

has the form

$$u(x, t) = f\left(t - \frac{x}{a}\right) .$$

In the case of a forced vibration which is determined by the time dependence through the factor $e^{i\omega t}$, a plane wave is described by

$$u(x, t) = Ae^{i(\omega t - kx)} , \tag{7-4.9}$$

where $k = \omega/a$ is the wave number and $|A|$ is the amplitude.

A plane wave which propagates in the direction $\boldsymbol{l}$ [where $\boldsymbol{l}$ (l_x, l_y, l_z) is a unit vector] can be described as follows:

$$u(x, y, z, t) = Ae^{i[\omega t - k(xl_x + yl_y + zl_z)]} = Ae^{i[\omega t - k\boldsymbol{l}\boldsymbol{r}]} . \tag{7-4.10}$$

The functions

$$v(x) = Ae^{-ikx} , \quad v(x, y, z) = Ae^{-ik\boldsymbol{l}\boldsymbol{r}} , \tag{7-4.11}$$

which are solutions of the vibration equation

$$\Delta v + k^2 v = 0 \tag{7-4.12}$$

likewise are called plane waves.

In the mathematical theory of diffraction, the excitation of a field produced by the insertion of inhomogeneities T_i in a homogeneous medium is investigated. Let $v(M)$ be the field in the homogeneous medium. This is produced by given sources which are distributed outside of the region T_i $(i = 1, \cdots, n)$; in particular, one can treat here sources which are sufficiently far from the observer and which lead to the occurrence of plane waves

$$\bar{v}(x, y, z) = Ae^{-ik\boldsymbol{l}\boldsymbol{r}} . \tag{7-4.13}$$

The actual field v_0 which is present in the region T_0 when inhomogenities exist can be represented in the form

$$v_0(M) = w_0(M) + \bar{v}_0(M) ,$$

where $\bar{v}_0(M)$ is the "the incident wave" and $w_0(M)$ is the diffracted or reflected wave. The latter represents the excitation of the external field $\bar{v}$ produced by the inhomogeneities T_i.

In the region T_0 we shall investigate the "diffracted" field $w_0(M)$, and inside T_i, the field v_i of the broken wave. First we give the conditions by which the functions w_0 and v_i $(i = 1, 2, \cdots, n)$ are determined.

(a) The functions w_0 and v_i satisfy the equations

$$\begin{aligned} \Delta w_0 + k_0^2 w_0 &= 0 \text{ in } T_0 , \\ \Delta v_i + k_i^2 v_i &= 0 \text{ in } T_i , \quad (i = 1, 2, \cdots, n) . \end{aligned} \tag{7-4.14}$$

(b) On the boundaries Σ_i of the region T_i and T_0 the following conditions are satisfied:

$$v_i = w_0 + \bar{v}_0 \quad \text{on } \Sigma_i \tag{7-4.15}$$

(v_0 is a prescribed function) and

$$p_i \frac{\partial v_i}{\partial \nu} = p_0 \frac{\partial w_0}{\partial \nu} + f_i \quad \text{on } \Sigma_i \tag{7-4.16}$$

[$f_i = p_0(\partial \bar{v}_0/\partial \nu_i)$ are prescribed functions].

(c) The reflected wave $w_0(M)$ at infinity behaves as a divergent spherical wave, that is, it satisfies the radiation conditions

$$w_0(M) = O\left(\frac{1}{r}\right), \quad \frac{\partial w_0}{\partial r} + ikw_0 = o\left(\frac{1}{r}\right).$$

2. Now we shall investigate more precisely the diffraction of a plane wave on a sphere.[31]

A plane wave

$$\bar{v} = Ae^{-ikz} \tag{7-4.17}$$

moves from infinity in the direction of the z-axis towards a sphere of radius R whose center is at the origin of coordinates. We expand the field of the reflected and diffracted wave in terms of spherical functions; the functions $\bar{v}_0$ and $f = p_0(\partial \bar{v}_0/\partial r)$ occurring on the right side of the above conditions are similarly expanded.

If we set $z = r\cos\theta$, then the following expansion of the plane wave in terms of spherical functions is valid:

$$e^{-ikr\cos\theta} = \sum_{m=0}^{\infty} (2m+1)(-i)^m \psi_m(kr) P_m(\cos\theta). \tag{7-4.18}$$

where

$$\psi_m(kr) = \sqrt{\frac{\pi}{2kr}} J_{m+1/2}(kr).$$

Thus $J_{m+1/2}(kr)$ is the Bessel function of the first kind of order $m + \frac{1}{2}$, and $P_m(kr)$ is the Legendre polynomial of the mth order. On the left side, we have a solution of the vibration equation which is dependent only on z. Each solution of the vibration equation, however, can be represented as the sum of products of spherical functions and the $\psi_m(kr)$. Since in our case the left side of (7-4.18) is dependent only on z, we have

$$e^{-ikr\cos\theta} = \sum_{m=0}^{\infty} C_m \psi_m(kr) P_m(\cos\theta), \tag{7-4.19}$$

where C_m are, first of all, still undetermined coefficients. With the use of the orthogonality of the Legendre polynomials and their norms (see Appendix A-3.8), we obtain

[31] Analogous methods have often been used in quantum mechanics for problems of the dispersion of particles.

$$C_m\psi_m(\rho) = \frac{2m+1}{2}\int_{-1}^{1} e^{-i\rho\xi} P_m(\xi)\, d\xi\,, \quad (\rho = kr,\ \xi = \cos\theta)\,. \tag{7-4.20}$$

Now we determine the first term in the asymptotic repesentation for the integral standing on the right side. By comparison of these members with the first term of the asymptotic expansion of the function $\psi_m(\rho)$, C_m can be determined. We obtain the expansion of the integral in powers of $1/\rho$, in which we partially integrate m times, where $e^{-i\rho\xi}$ is integrated and $P_m(\xi)$ is differentiated each time. Now if we consider only the first term of this expansion, we obtain

$$\begin{aligned}\int_{-1}^{+1} e^{-i\rho\xi} P_m(\xi)\, d\xi &\approx \frac{1}{-i\rho}[e^{-i\rho\xi} P_m(\xi)]_{-1}^{+1} = \frac{1}{-i\rho}(e^{-i\rho}P_m(1) - e^{i\rho}P_m(-1)) \\ &= \frac{1}{-i\rho}(e^{-i\rho} - (-1)^m e^{i\rho}) = \frac{1}{-i\rho}(e^{-i\rho} - e^{-im\pi}e^{i\rho}) \\ &= \frac{e^{-im(\pi/2)}}{-i\rho}\, e^{-i[\rho - m(\pi/2)]} - e^{i[\rho - m(\pi/2)]}] = 2(-i)^m \frac{\sin(\rho - (m\pi/2))}{\rho}\,.\end{aligned}$$

On the other hand, as is known, see (Appendix A-3.3(2))

$$\psi_m(\rho) \approx \frac{\sin(\rho - (m\pi/2))}{\rho}\,.$$

By comparing these expressions, we obtain from (7-4.20)

$$C_m = (2m+1)(-i)^m\,, \tag{7-4.21}$$

whereby formula (7-4.18) is proved.

From (7-4.17) it follows that

$$\begin{aligned}\bar{v}_0\,|_{r=R} &= \sum_{m=0}^{\infty} a_m P_m(\cos\theta)\,; \\ a_m &= A(2m+1)(-i)^m \psi_m(k_0R)\,;\end{aligned} \tag{7-4.22}$$

$$\begin{aligned}p_0 \frac{\partial \bar{v}_0}{\partial r}\bigg|_{r=R} &= \sum_{m=0}^{\infty} b_m P_m(\cos\theta)\,; \\ b_m &= Ak_0p_0(2m+1)(-i)^m \psi_m'(k_0R)\,.\end{aligned} \tag{7-4.23}$$

The field of the reflected wave and that of the broken wave are solutions of the wave equations. As in the incident field, they are dependent only on z. Therefore, we can write the functions v_1 aud w_0 in the form

$$v_1 = \sum_{m=0}^{\infty} \alpha_m \psi_n(k_1r) P_m(\cos\theta)\,, \tag{7-4.24}$$

$$w_0 = \sum_{m=0}^{\infty} \beta_m \zeta_m(k_0r) P_m(\cos\theta)\,, \tag{7-4.25}$$

and

$$\zeta_m(\rho) = \sqrt{\frac{\pi}{2\rho}}\, H_{m+1/2}^{(2)}(\rho)\,. \tag{7-5.26}$$

Now we turn to the determination of the expansion coefficients α_m and β_m.

By using conditions (7-4.3) and by comparing the coefficients of $P_m(\cos\theta)$, we obtain

$$\alpha_m\psi_m(k_1R) - \beta_m\zeta_m(k_0R) = a_m = A(2m+1)(-i)^m\psi_m(k_0R)\,,$$
$$p_1k_1\alpha_m\psi_m'(k_1R) - p_0k_0\beta_m\zeta_m'(k_0R) = b_m = Ak_0p_0(2m+1)(-i)^m\psi_m'(k_0R)\,,$$

and thus

$$\alpha_m = A(2m+1)(-i)^m\frac{p_0k_0[\psi_m(k_0R)\zeta_m'(k_0R) - \zeta_m(k_0R)\psi_m'(k_0R)]}{p_0k_0\psi_m(k_1R)\zeta_m'(k_0R) - p_1k_1\psi_m'(k_1R)\zeta_m(k_0R)}\,, \qquad (7\text{-}4.27)$$

$$\beta_m = A(2m+1)(-i)^m\frac{p_1k_1\psi_m(k_0R)\psi_m'(k_1R) - p_0k_0\psi_m'(k_0R)\psi_m(k_1R)}{p_0k_0\psi_m(k_1R)\zeta_m'(k_0R) - p_1k_1\psi_m'(k_1R)\zeta_m(k_0R)}\,. \qquad (7\text{-}4.28)$$

3. We now consider, as an example, the dispersion of sound waves by a solid spherical obstacle. The center of the solid and stationary sphere is at the origin of coordinates. The sphere is struck by a plane sound wave which propagates along the z-axis. As in Section 2-1, it can be shown that the sound pressure $p(x, y, z, t)$ satisfies the wave equation

$$\frac{\partial^2 p}{\partial t^2} = a^2\Delta p\,, \quad a^2 = \gamma\frac{p_0}{\rho_0}\,.$$

Here a is the sound velocity and γ is the adiabatic index; p_0 and ρ_0 are the pressure and density, respectively, of the medium in the undisturbed state.

The pressure carried by the incident plane wave is described by

$$\bar{p}_0 = Ae^{-i(\omega t - kz)}\,, \quad \left(k = \frac{\omega}{a}\right),$$

where A is a constant.

By analysis of the induced process

$$p(x, y, z, t) = p(x, y, z)e^{-i\omega t}\,,$$

we obtain for $p(x, y, z)$ the vibration equation

$$\Delta p + k^2p = 0\,.$$

Because of the rigidity of the sphere, the normal component of the velocity $\boldsymbol{u}$ must vanish on the spherical surface S_R. The projection of the velocity on the normal direction $\boldsymbol{n}$ is related to the pressure by

$$\frac{\partial u_n}{\partial t} = -\frac{1}{\rho}\frac{\partial p}{\partial n}\,.$$

In the stationary case, therefore, we obtain

$$u_n = \frac{1}{i\omega\rho}\frac{\partial p}{\partial n}\,,$$

from which we obtain the boundary condition

$$\left.\frac{\partial p}{\partial n}\right|_{S_R} = 0\,.$$

Now we set $p = p_0 + w$, where $w(x, y, z)$ denotes the pressure of the dispersed wave. Then for the determination of w there result the following conditions:

(a) The function $w(x, y, z)$ satisfies the vibration equation

$$\Delta w + k^2 w = 0 .$$

(b) On the spherical surface S_R, the boundary condition

$$\left.\frac{\partial w}{\partial n}\right|_{S_R} = - \left.\frac{\partial \bar{p}_0}{\partial n}\right|_{S_R}$$

holds.

(c) The dispersed wave w at infinity behaves like a divergent spherical wave, that is, for $r \to \infty$, w satisfies the radiation conditions

$$w(M) = O\left(\frac{1}{r}\right), \quad \frac{\partial w}{\partial r} + ikw = o\left(\frac{1}{r}\right) .$$

This problem is a special case of the diffraction problem considered above; indeed, it corresponds to the parameter value $p_1 = 0$.

If, in formulas (7-4.25) and (7-4.28), we now set $p_1 = 0$, then we obtain

$$\beta_m = - A(2m + 1)(- i)^m \frac{\psi_m'(k_0 R)}{\zeta_m'(k_0 R)} \tag{7-4.29}$$

and

$$w = - \sum_{m=0}^{\infty} (2m + 1)(- i)^m \frac{\psi_m'(k_0 R)}{\zeta_m'(k_0 R)} \zeta_m(k_0 r) P_m(\cos\theta) . \tag{7-4.30}$$

If the wave length is large in comparison with the size of the sphere, that is, if $k_0 R \ll 1$, then in formula (7-4.29) for the functions $\psi_m(kR)$ and $\zeta_m(kR)$, we can use a series expansion. From the expansion of the functions $J_{m+1/2}(kR)$ and $H^{(2)}_{m+1/2}(kR)$ in powers of the small argument kR (see Volume I, Appendix, Sections 1 and 3) we find:

$$\psi_0(kR) \approx \sqrt{\frac{\pi}{2kR}} \left(\frac{\left(\frac{1}{2}kR\right)^{1/2}}{\Gamma(\frac{3}{2})} - \frac{\left(\frac{1}{2}kR\right)^{5/2}}{\Gamma(\frac{5}{2})} \right), \quad \psi_1(kR) \approx \sqrt{\frac{\pi}{2kR}} \frac{\left(\frac{1}{2}kR\right)^{3/2}}{\Gamma(\frac{5}{2})} ,$$

$$\zeta_0(kR) \approx i \sqrt{\frac{\pi}{2kR}} \frac{\left(\frac{kR}{2}\right)^{-1/2}}{\Gamma(\frac{1}{2})} , \quad \zeta_1(kR) \approx - i \sqrt{\frac{\pi}{2kR}} \frac{\left(\frac{kR}{2}\right)^{-3/2}}{\Gamma(-\frac{1}{2})} .$$

Now, since

$$\Gamma(\tfrac{1}{2}) = 1\sqrt{\pi} , \quad \Gamma(\tfrac{3}{2}) = \frac{1\sqrt{\pi}}{2} , \quad \Gamma(\tfrac{5}{2}) = \tfrac{3}{4}1\sqrt{\pi} , \quad \Gamma(-\tfrac{1}{2}) = - 2\sqrt{\pi} ,$$

we obtain

$$\psi_0(kR) \approx 1 - \frac{(kR)^2}{6} , \quad \psi_1(kR) \approx \frac{kR}{3} ,$$

$$\zeta_0(kR) \approx \frac{i}{kR} , \qquad \zeta_1(kR) \approx \frac{i}{(kR)^2} ,$$

and from this we find

$$\psi_0'(kR) = -\frac{kR}{3}, \qquad \psi_1'(kR) \approx \frac{1}{3},$$

$$\zeta_0'(kR) \approx -\frac{i}{(kR)^2}, \quad \zeta_1'(kR) \approx \frac{2i}{(kR)^3}.$$

If we introduce the expression obtained for ψ_m' and ζ_m' into formula (7-4.29), then we obtain

$$\beta_0 = i\frac{A}{3}(kR)^3, \quad \beta_1 = -\frac{A}{2}(kR)^3;$$

We easily see that the resulting coefficients are proportional to $(kR)^5$. Therefore, for the dispersion of long waves ($kR \ll 1$), the excitation w can be described approximately by the first two terms of the series (7-4.30):

$$\begin{gathered} w \approx \beta_0\zeta_0(kr) + \beta_1\zeta_1(kr)\cos\theta, \\ [P_0(\cos\theta) = 1, \quad P_1(\cos\theta) = \cos\theta]. \end{gathered} \tag{7-4.31}$$

At large distances from the sphere ($kr \gg 1$), in the so-called wave zone, the asymptotic representations

$$\zeta_0(kr) \approx \frac{i}{kr}e^{-ikr}, \quad \zeta_1(kr) \approx -\frac{1}{kr}e^{-ikr} \tag{7-4.32}$$

hold for the functions $\zeta_0(kr)$ and $\zeta_1(kr)$. These formulas result from the asymptotic representations of the Hankel functions.

We now introduce expression (7-4.32) for $\zeta_0(kr)$ and $\zeta_1(kr)$ into formula (7-4.31), and replace β_0 and β_1 by their approximate values. Then we obtain

$$w \approx -\frac{Ak^2R^3}{3r}\left(1 - \frac{3}{2}\cos\theta\right)e^{-ikr}. \tag{7-4.33}$$

We turn now to the calculation of the intensity of the dispersed wave. This quantity is defined as the average value of the energy flux (Poynting vector), which is equal to the product of the excess sound pressure w and the velocity u, where w and u are the real parts of the corresponding expression. In our case,

$$\begin{aligned} w &= w_0\cos(\omega t - kr), \\ u &= u_0\cos(\omega t - kr), \end{aligned} \tag{7-4.34}$$

where w_0 and u_0 are the corresponding amplitudes.

For the calculation of the sound intensity I in the wave zone, we consider only the principal term of the asymptotic representations:

$$I = \frac{u_0w_0}{2T}\int_0^T \cos^2(\omega t - kr)\,dt = \frac{u_0w_0}{2}, \quad \left(T = \frac{2\pi}{\omega} \text{ is the period}\right).$$

From the equation of motion

$$\frac{\partial u}{\partial t} = -\frac{1}{\rho}\frac{\partial \omega}{\partial r}$$

and formula (7-4.34), we find

$$u_0 = \frac{w_0}{a\rho}.$$

Therefore,

$$I = \frac{w_0}{2a\rho} = \frac{A^2k^4R^6}{18a\rho r^2}\left(1 - \frac{3}{2}\cos\theta\right)^2 . \quad .$$

If we denote the amount dispersed by the sphere into the cone $d\theta$ by

$$2\pi r^2 \Sigma(\theta) \sin\theta \, d\theta ,$$

then we obtain

$$\Sigma(\theta) = \frac{A^2k^4R^6}{18a\rho}\left(1 - \frac{3}{2}\cos\theta\right)^2 .$$

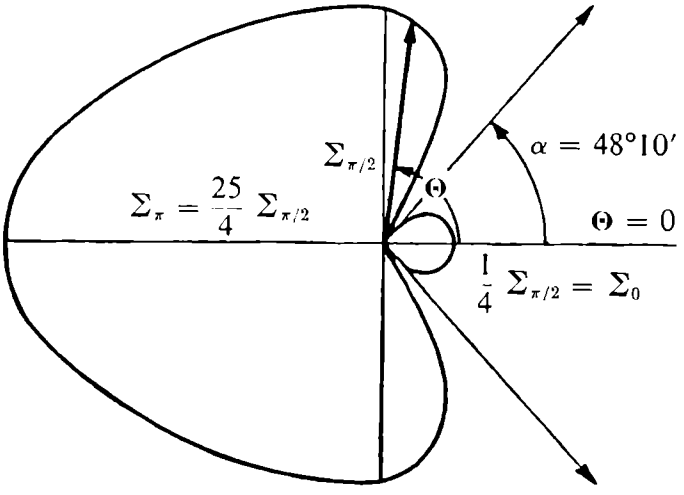


FIG. 89

Figure 89 shows the polar diagram for the intensity of a sound wave dispersed by a sphere (without consideration of the scale of measurement). If

$$\cos\theta = +\frac{2}{3}, \quad \theta = \alpha = 48°10' ,$$

then no dispersion occurs in the direction $\theta = \alpha$.

Problems on Chapter 7

1. Determine Green's function of a stationary point gas source under the assumption that the gas decomposes into a diffusion process. Solve the diffusion problem in space and in the plane.

2. Consider the same problem in the half space $y > 0$ under the assumption that the concentration is equal to zero for $y = 0$.

3(a) Solve the interior and exterior first boundary-value problems for the equation

$$\Delta u - \kappa^2 u = 0 ,$$

if the boundary condition

$$u|_{r=r_0} = A\cos\theta$$

is prescribed on the sphere of radius $r = r_0$.

In the case of the exterior problem, formulate conditions at infinity which guarantee the uniqueness of the solution.

Consider the corresponding problems for the boundary value.

$$u|_{r=r_0} = F(\theta)\,.$$

(b) Solve the corresponding problems for the two-dimensional equation. The boundary values here are prescribed on the circumference of the circle of radius $r = r_0$ and have the form

$$u|_{r=r_0} = A\cos\varphi$$

or

$$u|_{r=r_0} = F(\varphi)$$

4. Solve problems [3(a), (b)] for the equation

$$\Delta u + k^2 u = 0\,.$$

In the case of the interior problem, investigate for which value of r_0 there exists a unique solution (k is considered as prescribed).

Formulate conditions by which the uniqueness of the solution is guaranteed both for the case of two as well as for the case of three independent variables.

5. Under the surface of the earth at a depth h, a radioactive substance is found which is distributed there with a constant density. Determine the concentration of the emanation under the assumption that its concentration on the surface is equal to zero.

6. Determine the eigenvibrations of a ring-shaped membrane. Let the radii of the rings be a and b ($a < b$).

Let $v|_{r=a} = 0$ and $v|_{r=b} = 0$. Show that the limit value of the first eigenvalue for $a \to 0$ is equal to the first eigenvalue of the circular membrane of radius b with a fixed boundary.

7. Determine the eigenvibrations and the eigenfrequencies of an endovibrator of cylindrical form under the assumption that the walls of the endovibrator are ideal conductors. Consider the same problem for the acoustics case.

Hint: In the case of electromagnetic vibrations we introduce a polarization potential (see Section 7-5.1).

8. Determine the electromagnetic field of a point dipole in an unbounded space under the assumption that the size of the field is proportional to $e^{i\omega t}$. Investigate the asymptotic ratio of the solution at large distances (in the wave zone). Solve the same problem for a dipole which lies on a perfectly conducting surface (vertical dipole).

Hint: Use the polarization potential.

9. Formulate the problem of propagation of electromagnetic waves within an infinitely long cylindrical cavity of arbitrary cross section with perfectly

conducting walls. Consider the electrical wave which propagates along a circular cylindrical cavity and has the greatest length. Determine the field and calculate the energy flux through a cross section which is perpendicular to the generators (of the cylinder) (see Section 7-5.1).

10. Solve the inhomogeneous equation

$$\Delta u + k^2 u = -f$$

in an unbounded cylindrical region of circular cross section. On the surface of this region, homogeneous boundary conditions of the first or second kind are to be satisfied. Construct the Green's function (see Section 7-5.2).

Construct Green's function in the case of the first boundary-value problem for the equation

$$\Delta u + k^2 u = 0$$

(a) In the half space $z > 0$, (b) In the half plane $y > 0$, (c) In the layer $l \leqq z \leqq l$.

12. Solve the problem of the diffraction of a plane electromagnetic wave by a perfectly conducting cylinder. Consider the same problem for the acoustics case.

13. Determine the electromagnetic eigenvibrations of a spherical endovibrator with perfectly conducting walls. In particular, consider the vibrators of the type *TE* and *TM*.

14. Determine the electromagnetic eigenvibrations of an endovibrator which describes a region bounded between two coaxial cylindrical surfaces and two planes which are perpendicular to the cylinder axis (Figure 90).

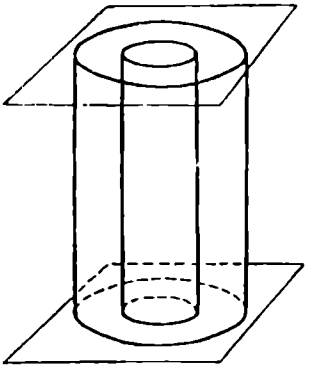


FIG. 90

Hint: For the polarization potential $Z_{n,m}$ use a formula which is analogous to formula (7-5.14).

7-5. APPLICATIONS OF CHAPTER 7

1. Waves in conducting tubes (wave guides)

1. For the construction of different types of shortwave radio transmitters, we are faced with the important problem of the transport of electromagnetic

energy from the sender to the radiating antenna or from the antenna to the receiver. The question of transport of electromagnetic energy also occurs for a series of other important practical problems of modern radio technology.

Until a few years ago these problems were solved satisfactorily in that the transport was accomplished with the help of double conductors. Here we have two metallic conductors between which electromagnetic waves propagate. In this case, however, it is apparent that, aside from the defects which generally accompany all propagating conductors, such double conductors always radiate a fixed amount of electromagnetic energy, and the higher the frequency of the propagating wave, the greater the radiation. On this basis, propagation with the help of double conductors in the region of ultrashort waves is unsuitable.

In recent years, in the technology of ultrashort waves (centimeter and decimeter waves) entirely different means of propagation for the transport of energy have been developed. These consist of metallic tubes (conducting tubes) inside of which radio waves propagate. Such hollow conductors show very small losses, so that they are especially suitable as transport conductors.[32]

The mathematical theory of the propagation of radio waves in hollow conductors was already established by Rayleigh. Rayleigh investigated the propagation of acoustic waves in tubes. In recent years, the theory of hollow conductors (wave guides) has been developed extensively, particularly by Soviet investigators. The properties of circular, rectangular and other types of hollow conductors are also under investigation.†

We will consider first the properties of hollow conductors of arbitrary cross section and then illustrate these properties with a series of concrete examples. Taking a cylindrical hollow conductor unbounded along the z-axis, we assume its wall to be a perfect conductor. Let Σ be the surface, S the cross section, and C the contour which bounds this cross section. Further we assume: (a) the dielectric constant ε and the magnetic permeability μ of the medium filling the tube is equal to one. Further, let $\sigma = 0$. (b) Let the region inside of the hollow conductor be source free. (c) Let the field vary periodically according to $e^{-i\omega t}$.

With these assumptions, Maxwell's equations have the form

$$\begin{aligned} &\text{rot } \boldsymbol{H} = -ik\boldsymbol{E}\,, \\ &\text{rot } \boldsymbol{E} = ik\boldsymbol{H}\,, \quad (k = \omega/c) \\ &\text{div } \boldsymbol{H} = 0\,, \\ &\text{div } \boldsymbol{E} = 0\,. \end{aligned} \tag{7-5.1}$$

Since the walls of the conductor are assumed to be perfectly conducting, the tangential component E_t directly on the wall of the conductor is equal to zero:

$$E_t|_\Sigma = 0\,. \tag{7-5.2}$$

[32] B. A. Vvedenskii and A. G. Arenberg, *Radio Waves Guides*, Part I, Moscow, 1946.

† Editor's note: In particular, the work of L. A. Vainstein is to be noted. SeeNew York Univ. Report No. EM-63, Institution of Mathematical Sciences, Div. of Electromagnetic Res., Jan. 1954 (Transl. by J. Shmoys).

Next we show that inside the conductor, progressive electromagnetic waves can propagate.

We seek the solution of Eq. (7-5.1) in the form

$$\begin{aligned} \boldsymbol{E} &= \operatorname{grad}\operatorname{div}\mathfrak{Z} + k^2\mathfrak{Z}\,, \\ \boldsymbol{H} &= -ik\operatorname{rot}\mathfrak{Z}\,, \end{aligned} \tag{7-5.3}$$

where $\mathfrak{Z}$ is the Herz vector (the polarization potential). Here we consider the case in which $\mathfrak{Z}$ possesses only one component in the direction of the z-axis. In this case, after inserting form (7-5.3), we obtain from Eq. (7-5.1):

$$\Delta Z + k^2 Z = 0 \quad \text{or} \quad \Delta_2 Z + \frac{\partial^2 Z}{\partial z^2} + k^2 Z = 0 \tag{7-5.4}$$

$$(\mathfrak{Z} = Z\boldsymbol{i}_z)\,.$$

The boundary condition (7-5.2) is satisfied if we require that

$$Z|_\Sigma = 0\,. \tag{7-5.5}$$

For Z we use the Ansatz

$$Z(M, z) = \psi(M)f(z)\,, \tag{7-5.6}$$

where M denotes a point lying in the cross section S. If we introduce expression (7-5.6) into Eq. (7-5.4), then we find that $\psi(M)$ is an eigenfunction of the problem

$$\begin{aligned} &\Delta_2\psi + \lambda\psi = 0 \quad \text{in the interior of } S \\ &\psi|_C = 0\,. \end{aligned} \tag{7-5.7}$$

This problem corresponds to the vibrations of a membrane which is fixed on the boundary; $\Delta_2 = (\partial^2/\partial x^2) + (\partial^2/\partial y^2)$ is the two dimensional Laplace operator.

We denote the eigenvalues and eigenfunctions of this problem by $\{\lambda_n\}$ and $\{\psi_n\}$. Special solutions of Eq. (7-5.4) are then the functions

$$Z_n(M, z) = \psi_n(M)f_n(z)\,,$$

where the functions $f_n(z)$ are integrals of the equation

$$f_n'' + (k^2 - \lambda_n)f_n = 0\,. \tag{7-5.8}$$

The general solution of Eq. (7-5.8) is

$$f_n(z) = A_n e^{i\gamma_n z} + B_n e^{-i\gamma_n z}\,, \quad (\gamma_n = \sqrt{k^2 - \lambda_n})\,. \tag{7-5.9}$$

As is easily seen, the term $A_n e^{i\gamma_n z}$ corresponds to a wave propagating in a positive z direction, and the second term in Eq. (7-5.9) corresponds to a wave propagating in the reverse direction.

If we consider a wave propagating only in one direction, then we obtain

$$f_n(z) = A_n e^{i\gamma_n z}$$

so our solutions have the form

$$Z_n(M, z) = A_n\psi_n(M)f_n(z)\,. \tag{7-5.10}$$

The constants A_n are calculated from the excitation conditions of the field.

By substituting Eq. (7-5.10) into (7-5.3) and extracting the factor $e^{-i\omega t}$, we obtain

$$Z_n(M, z) = F_n(M)e^{i(\gamma_n z - \omega t)}, \tag{7-5.11}$$

where F_n are functions which are expressed through the eigenfunctions $\psi_n(M)$ of the membrane or their derivatives.

If $k^2 > \lambda_n$, then γ_n is real and expression (7-5.11) represents a propagating wave which propagates along the z-axis with phase velocity

$$v = \frac{\omega}{\sqrt{k^2 - \lambda_n}} = \frac{c}{\sqrt{1 - (\lambda_n / k^2)}} > c .$$

The group velocity of the wave is obviously equal to

$$u = \frac{c^2}{v}\sqrt{1 - \frac{\lambda_n}{k^2}} < c ,$$

that is, in the inner space of the hollow conductor, dispersion occurs.

If $k^2 < \lambda_n$, then $\gamma_n = i\kappa_n$ ($\kappa_n > 0$) and we obtain in place of expression (7-5.11) the damped wave

$$F_n(M)e^{-i\omega t - \kappa_n z}, \tag{7-5.12}$$

which propagates along the z-axis in the positive direction.

Since the eigenfrequencies λ_n of the membrane increase unboundedly with increasing n, then independent of the frequency ω, from a fixed number $n = N$ on,

$$k^2 < \lambda_n$$

holds. Consequently, in a hollow conductor only a finite number of progressive waves can propagate. Therefore, if $k^2 < \lambda_1$, then in the hollow conductor there exists no single propagating wave.

Thus, in order for a wave to propagate at all in a hollow conductor of a given form and size, obviously the condition

$$\lambda_1 < k^2 \quad \text{or} \quad l < \frac{2\pi}{\sqrt{\lambda_1}}$$

must be satisfied, where l is the length of the propagating wave in the hollow conductor.

For a hollow conductor with a rectangular cross section (a and b are the lengths of the sides) the following holds:

$$\lambda_n = \lambda_{mn} = \pi^2\left(\frac{m^2}{a^2} + \frac{n^2}{b^2}\right), \quad (m = 1, 2, \cdots; \; n = 1, 2, \cdots) . \tag{7-5.13}$$

Therefore, only if

$$k > \pi\sqrt{\frac{1}{a^2} + \frac{1}{b^2}} \quad \text{or} \quad l < \frac{2}{\sqrt{\frac{1}{a^2} + \frac{1}{b^2}}} \tag{7-5.14}$$

can a propagating wave exist.

For solutions of the Maxwell's equations, one should also consider fields for which the z component of the electrical field is equal to zero:

$$E_z = 0 . \tag{7-5.15}$$

If now we introduce the vector $\hat{3} = \hat{Z}\boldsymbol{i}_z$ and set

$$\boldsymbol{E} = ik \operatorname{rot} \hat{3} , \quad \boldsymbol{H} = \operatorname{grad} \operatorname{div} \hat{3} + k^2 \hat{3} . \quad (\hat{E}_z = 0) , \tag{7-5.16}$$

then we recognize that the function $\hat{Z}(M, z)$ must satisfy the equation

$$J\hat{Z} + k^2 \hat{Z} = 0 \quad \text{or} \quad J_2 \hat{Z} + \frac{\partial^2 \hat{Z}}{\partial z^2} + k^2 \hat{Z} = 0 \tag{7-5.17}$$

and the boundary condition

$$\frac{\partial \hat{Z}}{\partial \nu} = 0 \quad \text{on } \Sigma . \tag{7-5.18}$$

By repetition of these arguments, we find the solutions

$$\hat{Z}_n = \hat{A}_n \hat{\varphi}_n(M) e^{i\hat{\gamma}_n z} , \quad (\hat{\gamma}_n = \sqrt{k^2 - \hat{\lambda}_n}) \tag{7-5.19}$$

of the problem which as solutions of Maxwell's equations correspond to the functions

$$\hat{F}_n(M) e^{i(\hat{\gamma}_n z - \omega t)} .$$

Here $\hat{\varphi}_n(M)$ and $\hat{\lambda}_n$ are eigenfunctions and eigenfrequencies of the membrane S with a free boundary, that is, the eigenvalue problem

$$J_2 \hat{\varphi}_n + \hat{\lambda}_n \hat{\varphi}_n = 0 \quad \text{in } S$$

$$\frac{\partial \hat{\varphi}_n}{\partial \nu} = 0 \qquad \text{on } C .$$

Accordingly, in a hollow conductor, electromagnetic fields of the type $\{\boldsymbol{E}, \boldsymbol{H}\}$ and $\{\hat{\boldsymbol{E}}, \hat{\boldsymbol{H}}\}$ can occur, as we have established by formulas (7-5.3) and (7-5.16). The following terminology is common: We speak of electrical waves (or of waves of type TM), provided $H_z = 0$, or of magnetic waves (type TE), provided $E_z = 0$. We say that in a hollow conductor waves not only of type TE but also of type TM can occur. We can show that an arbitrary field in an hollow conductor can always be represented as a sum of fields of types TE and TM.[33]

Thus it follows that an arbitrary field in a hollow conductor can be determined if both scalar functions $Z(M, z)$ and $\hat{Z}(M, z)$ are known.

2. We shall now determine the energy which a propagating wave of type TM possesses.

For this we calculate the magnitude of the flux of the Poynting vector through the cross section S:

$$W_z = \frac{c}{8\pi} \int_S \int [\boldsymbol{E}\boldsymbol{H}^*]_z \, dS . \tag{7-5.20}$$

[33] A. N. Tychonov and A. A. Samarski, *Vestnik Moscow University* 7, 1948.

Here $\boldsymbol{H}^*$ is the complex conjugate vector to $\boldsymbol{H}$ and S is the perpendicular cross section of the hollow conductor.

By using a rectangular coordinate system x, y, z, we have

$$W_z = \frac{c}{8\pi}\int_S\int (E_x H_y^* - E_y H_x^*)\, dx\, dy\ . \tag{7-5.21}$$

The field components are expressed through the polarization potential Z as follows:

$$E_x = \frac{\partial^2 Z}{\partial x \partial z}\ , \qquad E_y = \frac{\partial^2 Z}{\partial y \partial z}\ ,$$

$$H_x^* = ik\frac{\partial Z^*}{\partial y} \qquad H_y^* = -ik\frac{\partial Z^*}{\partial x}\ .$$

By substituting these expressions into Eq. (7-5.21), we obtain

$$W_z = -\frac{c}{8\pi}ik\int_S\int\left(\frac{\partial^2 Z}{\partial x \partial z}\frac{\partial Z^*}{\partial x} + \frac{\partial^2 Z}{\partial y \partial z}\frac{\partial Z^*}{\partial y}\right) dx\, dy\ . \tag{7-5.22}$$

The function Z and the complex function Z^* conjugate to it can be represented in the form

$$Z\ (M, z) = A_n \psi_n(M) e^{i\gamma_n z}$$
$$Z^*(M, z) = A_n^* \psi_n(M) e^{-i\gamma_n z}\ ,$$

where ψ_n are the eigenfunctions of a membrane fixed along the boundary $C(\psi_n|_C = 0)$. From this it follows that in place of (7-5.22)

$$W_z = \frac{ck}{8\pi}\gamma_n |\, A_n\,|^2 \int_S\int\left[\left(\frac{\partial\psi_n}{\partial x}\right)^2 + \left(\frac{\partial\psi_n}{\partial y}\right)^2\right] dx\, dy = \frac{ck}{8\pi}\gamma_n |\, A_n\,|^2 \int_S (\Gamma_2\psi_n)^2\, dS$$

also can be written. By application of the first Green's formula

$$\int_S\int (\Gamma_2\psi_n)^2\, dS = -\int_S\int \psi_n \Delta_2\psi_n\, dS + \int_C \psi_n\frac{\partial\psi_n}{\partial\nu}\, ds = \lambda_n\int_S\int\psi_n^2\, dS = \lambda_n\ ,$$

we finally obtain

$$W_z = \frac{ck}{8\pi}\,|\, A_n\,|^2\gamma_n\lambda_n \tag{7-5.23}$$

for the energy flux of the nth wave. If several waves propagate at the same time, then W_z is equal to the sum of all the summands of the form (7-5.23).

We now treat the problem of the generation of an electromagnetic field in a hollow conductor by a prescribed current.[34]

3. In a spatial region V_0 within a hollow conductor Σ, let there be a current $\boldsymbol{j}(M, z)e^{-i\omega t}$; the time dependence of the current is, therefore, harmonic. We seek the field generated by this current. On the basis of the superposition principle, it is obviously sufficient to consider the excitation in hollow conductors by an elementary dipole of arbitrary orientation.

In order to give an idea of the method for the solution of these general

[34] A. A. Samarski and A. N. Tychonov, *Journal fur theoretische Physik*, 27, 11, 12, 1947.

problems, we first consider the simplest case of the excitation of the hollow conductor by a linear current $I = I_0(z)e^{-i\omega t}$ which is prescribed on an interval L parallel to the z-axis.

For the determination of the electromagnetic field developed in the hollow conductor, we should consider:

(a) Maxwell's equations (7-5.1);

(b) The boundary condition

$$E_{\tan} = 0 \text{ on } \Sigma ;$$

(c) The radiation conditions in the form of the requirement that no waves coming from infinity are present;

(d) The excitation condition which we assume in the form[35]

$$\oint_{K_\varepsilon} H_s \, ds = \frac{4\pi}{c} I_0 \quad \text{or} \quad H_s \approx \frac{2I_0}{c\rho} , \tag{7-5.24}$$

where K_ε denotes a circle of radius ε ($\varepsilon \to 0$) containing the interval L in its interior and $\rho = MM_0$ (M_0 is a point on L and M is a point on the circle K_ε). Consequently, the electromagnetic field must have a singularity of definite form on the current carrying line L.

We again use the polarization potential Z introduced through Eq. (7-5.3). Let (M_0, ξ) be an arbitrary point of the current carrying line. By the use of cylindrical coordinates ρ, ϕ, z, whose origin can lie at the point (M_0, ξ), we obtain from Eq. (7-5.3)

$$H_s = ik \frac{\partial Z}{\partial \rho} .$$

From this and from Eq. (7-5.24) it follows that Z at the point (M_0, ξ) must have a logarithmic singularity defined by

$$Z \approx \frac{2I_0}{-ikc} \ln \frac{1}{\rho} . \tag{7-5.25}$$

The function $Z(M, z)$ must, therefore, satisfy the vibration equation (7-5.4), the boundary condition $Z = 0$ on Σ, the radiation condition, and requirement (7-5.25).

Now for Z we make the Ansatz

$$Z = K \int_L Z_0(M, M_0; z, \zeta) I_0(\zeta) \, d\zeta , \tag{7-5.26}$$

where $Z_0(M, M_0; z, \zeta)$ is the corresponding Green's function which is determined as the solution of the equation

$$\Delta Z_0 + k^2 Z_0 = 0 ,$$

with respect to the variables (M, z) and (M_0, ζ), and which satisfies the boundary condition

$$Z_0 = 0 \quad \text{on } \Sigma ,$$

[35] See Eq. (5-4.6′)

as well as the radiation condition. There, where its arguments coincide, it possesses a singularity of the form $(1/4\pi)e^{ikr}/r$. The function Z_0, consequently, can be represented in the form

$$Z_0(M, M_0; z, \zeta) = \frac{e^{ikr}}{4\pi r} + v(M, M_0; z, \zeta)$$

$$(r = \sqrt{\rho^2 + (z-\zeta)^2},\quad \rho = MM_0),$$

where v is a regular function satisfying the vibration equation and the boundary condition

$$v = \frac{e^{ikr}}{4\pi r} \quad \text{on } \Sigma.$$

As is easily seen, the function $Z(M, z)$ defined by (7-5.26) has the required logarithmic singularity and satisfies the condition (7-5.25) if we consider the normalization factor

$$K = \frac{4\pi}{-ikc}.$$

Using this factor we obtain

$$Z(M, z) = \frac{4\pi}{-ikc}\int_L Z_0(M, M_0; z, \zeta) I_0(\zeta)\, d\zeta.$$

In particular, for a current element of length l we find

$$Z(M, z) = \frac{4\pi}{ikc} I \cdot l \cdot Z_0.$$

Consequently, Z_0 signifies a a polarization potential. This corresponds to the excitation by a current element which lies at the point (M_0, ξ) and flows parallel to the axis of the hollow conductor.

Therefore, the problem, that is, the determination of the field in a hollow conductor, is chmpletely reduced to the construction of the Green's function Z_0 of the first boundary-value problem for the equation $\Delta u + k^2 u = 0$ in an unbounded region.

For the construction of the Green's function, the method described in Section 6 2 can be used. To do this we consider the inhomogeneous equation

$$\Delta u + k^2 u = f(M, z) \tag{7-5.27}$$

(Let $f(M, z)$ be a given function) with the boundary condition

$$u|_\Sigma = 0.$$

For the function $u(M, z)$ which we seek, we make the series Ansatz

$$u(M, z) = \sum_{n=1}^{\infty} u_n(z)\varphi_n(M). \tag{7-5.28}$$

Here the $\varphi_n(M)$ are the normalized eigenfunctions for the membrane S [from (7 5.7)]:

$$\Delta_2\varphi_n + \lambda_n\varphi_n = 0,\quad \varphi_n|_C = 0.$$

Now, if we also expand $f(M, z)$ in a series of these eigenfunctions

$$f(M, z) = \sum_{n=1}^{\infty} f_n(z)\psi_n(M), \quad f_n(z) = \int_S \int f(M', z)\psi_n(M')\, d\sigma_{M'}, \qquad (7\text{-}5.29)$$

and introduce expressions (7-5.28) and (7-5.29) in (7-5.27), then we obtain

$$u_n''(z) - p_n^2 u_n(z) = -f_n(z), \quad p_n = \sqrt{\lambda_n - k^2}. \qquad (7\text{-}5.30)$$

The solution of this equation, as is easily proved, can be represented by the formula

$$u_n(z) = \int_{-\infty}^{+\infty} \frac{e^{-p_n|z-\zeta|}}{2p_n} f_n(\zeta)\, d\zeta. \qquad (7\text{-}5.31)$$

which can also be written in the form

$$u_n(z) = \iint_S \int_{-\infty}^{+\infty} \frac{e^{-p_n|z-\zeta|}}{2p_n} f(M', \zeta)\psi_n(M')\, d\sigma_{M'}\, d\zeta \qquad (7\text{-}5.31')$$

because of expression (7-5.29). By substituting these expressions in (7-5.28) and by an interchange of the order of summation with integration, we obtain

$$u(M, z) = \iint_T \int Z_0(M, M', z-\zeta) f(M', \zeta)\, d\sigma_{M'}\, d\zeta, \qquad (7\text{-}5.32)$$

where

$$Z_0(M, M', z-\zeta) = \sum_{n=1}^{\infty} \frac{\psi_n(M)\psi_n(M')}{2p_n} e^{-p_n|z-\zeta|}. \qquad (7\text{-}5.33)$$

The series for $Z_0(M, M', z-\zeta)$ converges for $z-\zeta$ uniformly and absolutely. This follows from the estimates for the eigenfunctions[36] and the presence of the exponential factors $e^{-p_n|z-\zeta|}$. For $M = M'$ and $z = \zeta$, the function $Z(M, M', z-\zeta)$ has a singularity of type $1/r$. We shall not go into the proof of this last statement.[37]

From the above statements, it follows that

$$G(M, M', z-\zeta) = Z_0(M, M', z-\zeta),$$

[36] For the eigenfunctions $\psi_n(M)$, the relation $|\psi_n(M)| \leq A\lambda_n$ holds uniformly, where A is a constant which does not depend either on the point M or on the index n.

Problem (7-5.7) is equivalent to the integral equation

$$\psi_n(M) = \lambda_n \int_S \int G(M, M')\psi_n(M')\, d\sigma_{M'}$$

in which $G(M, M')$ denotes Green's function of Laplace's equation $\Delta_2 u = 0$ with the boundary condition $u_C = 0$. From this integral equation, however, on the basis of the Schwarz inequality, we can obtain

$$|\psi_n| \leq |\lambda_n| \sqrt{\int_S \int G^2(M, M')\, d\sigma_{M'} \int_S \int \psi_n^2(M')\, d\sigma_{M'}} \leq A|\lambda_n|,$$

since

$$\int_S \int \psi_n^2(M')\, d\sigma_{M'} = 1, \quad \int_S \int G^2(M, M')\, d\sigma_{M'} \leq A^2.$$

For the derivatives, we obtain

$$\left|\frac{\partial \psi_n}{\partial x}\right| \leq B\lambda_n^2, \quad \left|\frac{\partial \psi_n}{\partial y}\right| \leq B\lambda_n^2.$$

correspondingly.

[37] See A. A. Samarski and A. N. Tychonov, *Journal für theoretische Physik*, 27, 11, 1947.

that is, Green's function Z_0 has the form

$$Z_0(M, M', z-\zeta) = \sum_{n=1}^{\infty} \frac{\psi_n(M)\psi_n(M')}{2p_n} e^{-p_n|z-\zeta|}.$$

From expression (7-5.33) we find that the field in this case can be represented by a superposition of waves of the forms (7-5.11) and (7-5.12). On the basis of the observation on p. 482, the series (7-5.33) consists of finitely many summands of the form

$$B_n\psi_n(M)e^{i\gamma_n|z-\zeta|}, \quad (\gamma_n = \sqrt{k^2-\lambda_n},\ p_n = -i\gamma_n) \quad \text{(progressing waves}$$

and infinitely many summands of the form

$$B_n'\psi_n(M)e^{-p_n|z-\zeta|}, \qquad \text{decaying waves)}$$

where

$$B_n' = \frac{\psi_n(M')}{2p_n}, \quad p_n = \sqrt{\lambda_n - k^2}, \quad \lambda_n > k^2.$$

For the determination of the field, formulas (7-5.26) and (7-5.3) are used.

The problem of excitation in hollow conductors by an element of magnetic current parallel to the z-axis (an infinitesimally small current-conducting loop in the plane $S_{z=\zeta}$) leads to the second Green's function

$$\hat{Z}_0(M, M', z-\zeta) = \sum_{n=1}^{\infty} \frac{\hat{\psi}_n(M)\hat{\psi}_n(M')}{2\hat{p}_n} e^{-\hat{p}_n|z-\zeta|},$$

$$\hat{p}_n = \sqrt{\hat{\lambda}_n - k^2},$$

which satisfies the boundary condition $\partial\hat{Z}_0/\partial\nu = 0$ on Σ. Here $H_z = 0$, $\hat{Z} = [1/(4\pi i\omega)]KI\hat{Z}_0$ (KI is the moment of the magnetic current).

In a similar way, we treat the problem of excitation of an arbitrarily oriented dipole (current elements) by determining, in this case, the existing field singularities. The corresponding functions Z are calculated from a formula which is analogous to formula 7-5.33. In the case of plane and spatial currents, the functions Z are surface and spatial integrals [by analogy with (7-5.26)]. Further calculations of the field follow from formula 7-5.16.

Thus the problem of excitation of an arbitrary cylindrical hollow conductor by an arbitrarily prescribed current is completely solved. In order to be able to use the general formulas for a hollow conductor with a definite cross section, it suffices to determine the eigenvibrations of that membrane which has the form of the perpendicular cross section of the hollow conductor.

The orthonormal eigenfunctions of a rectangular membrane with sides of length a and b are

$$\psi_n(M) = \psi_{mn}(x, y) = \sqrt{\frac{4}{ab}} \sin\frac{\pi m}{a}x \sin\frac{\pi n}{b}y,$$

$$\hat{\psi}_n(M) = \hat{\psi}_{mn}(x, y) = \sqrt{\frac{\varepsilon_m\varepsilon_n}{ab}} \cos\frac{\pi m}{a}x \cos\frac{\pi n}{b}y, \quad \varepsilon_j = 2,\ j \neq 0;\ \varepsilon_0 = 1,$$

$$\lambda_{mn} = \pi^2\left(\frac{m^2}{a^2} + \frac{n^2}{b^2}\right).$$

For a circular membrane of radius a, we have

$$\psi_n(M) = \psi_{mn}(r, \varphi) = \sqrt{\frac{\varepsilon_n}{\pi a^2}} \frac{J_n(\mu_{mn}(r/a))}{J_n'(\mu_{mn})} \begin{matrix}\cos \\ \sin\end{matrix} n\varphi ,$$

$$\hat{\psi}_n(M) = \hat{\psi}_{mn}(r, \varphi) = \sqrt{\frac{\varepsilon_n}{\pi a^2}} \frac{\hat{\mu}_{mn}}{\sqrt{\hat{\mu}_{mn}^2 - n^2}} \frac{J_n((\hat{\mu}_{mn}/a)r)}{J_n(\hat{\mu}_{mn})} \begin{matrix}\cos \\ \sin\end{matrix} n\varphi .$$

Thus μ_{mn} are the roots of the equations $J_n(\mu) = 0$; $\lambda_{mn} = \mu_{mn}^2/a^2$; $\hat{\mu}_{mn}$ denote the roots of $J_n'(\mu) = 0'$; $\lambda_{mn} = \hat{\mu}_{mn}^2/a^2$.

2. Electromagnetic vibrations in cavity resonators

In recent years, in radio technology the so-called endovibrators (cavity resonators) have found many uses. An endovibrator consists of a hollow metal body filled with a dielectric (in particular, with air). In such endovibrators, stationary electric fields (standing waves) can exist, that is, the so-called electromagnetic eigenvibrations.

In the technology of ultrashort waves, endovibrators of very complex forms are used. The general problem, to determine the eigenvibrations of endovibrators of arbitrary form, is extremely difficult; however, for simpler forms of endovibrators the explicit solution can be given. Since the walls of the endovibrators can be constructed from good conducting metals, they are usually assumed to be perfectly conducting for the calculation of the eigenvibrations. In the following discussion we assume this and that all quantities vary with time according to $e^{-i\omega t}$. The consideration of finite conductivities proceeds by using the boundary condition of Leontovitch.

We shall not include here a presentation of the theory of endovibrators, but shall discuss only certain general questions of the theory of these vibrating systems.

1. *The eigenvibrations of cylindrical endovibrators.* If we want to determine the electromagnetic eigenvibrations, we have to find the nontrivial solutions of Maxwell's equations.[38] More precisely, we must find those eigenfrequencies ω for which the system of homogeneous Maxwell's equations possess nontrivial solutions, and we must also determine the nontrivial solutions themselves.

In the interior of the cavity T, Maxwell's equations have the form

$$\begin{aligned} &\operatorname{rot} \boldsymbol{H} = -ik\boldsymbol{E} \\ &\operatorname{rot} \boldsymbol{E} = ik\boldsymbol{H}, \quad (k = \omega/c) \\ &\operatorname{div} \boldsymbol{E} = 0 \\ &\operatorname{div} \boldsymbol{H} = 0 . \end{aligned} \tag{7-5.34}$$

On the boundary Σ, the condition

$$E_t = 0 \tag{7-5.35}$$

or

$$\frac{\partial H_\nu}{\partial \nu} = 0 \tag{7-5.36}$$

[38] The factor $e^{-i\omega t}$ is omitted everywhere.

must be fulfilled; these two conditions, as can be easily shown, are equivalent.

We shall carry out the calculation of eigenvibrations for an endovibrator which is described by a section of a cylindrical hollow conductor of arbitrary cross section and is bounded on the two ends $z = +l$ (let the z-axis be parallel to the generators of the cylinder).

As in a cylindrical hollow conductor, it is also true in the endovibrators considered that vibrations of electrical type ($H_z = 0$) and vibrations of the magnetic type ($E_z = 0$) are possible.

For waves of the electrical type, we set

$$\begin{aligned} \boldsymbol{E} &= \operatorname{grad}\operatorname{div}\mathfrak{Z} + k^2\mathfrak{Z}\,, \\ \boldsymbol{H} &= -ik\operatorname{rot}\mathfrak{Z}\,, \end{aligned} \tag{7-5.37}$$

where $\mathfrak{Z} = Z\boldsymbol{i}_z = (\boldsymbol{i}_z$ is the unit vector parallel to the z-axis) is the polarization potential for which only the z component is different from zero. From (7-5.37) we note that in this case $H_z = 0$.

The function Z again satisfies the vibration equation

$$\Delta Z + k^2 Z = 0\,. \tag{7-5.38}$$

We now choose on the surface Σ a system of rectangular coordinates $(\boldsymbol{s}, \boldsymbol{\nu}, \boldsymbol{i}_z)$ which depends on the point. Here $\boldsymbol{\nu}$ denotes the unit vector in the direction of the surface normal at the point in question, and $\boldsymbol{s}$ is the unit vector in the direction of the tangent to the contour C. (C denotes the curve bounding the perpendicular crosssection S of the cylindrical endovibrator.)

On the basis of boundary condition (7-5.35), we have

$$\begin{aligned} E_s|_\Sigma &= \left.\frac{\partial^2 Z}{\partial s \partial z}\right|_\Sigma = 0\,, \\ E_z|_\Sigma &= \left.\left(\frac{\partial^2 Z}{\partial z^2} + k^2 Z\right)\right|_\Sigma = 0\,. \end{aligned} \tag{7-5.39}$$

Both these equations are satisfied if

$$Z|_\Sigma = 0\,. \tag{7-5.40}$$

For $z = -l$, we obtain from (7-5.35) the conditions

$$\begin{aligned} E_s|_{z=\pm l} &= \left.\frac{\partial^2 Z}{\partial s \partial z}\right|_{z=\pm l} = 0\,, \\ E_\nu|_{z=\pm l} &= \left.\frac{\partial^2 Z}{\partial \nu \partial z}\right|_{z=\pm l} = 0\,. \end{aligned}$$

These are satisfied if

$$\left.\frac{\partial Z}{\partial z}\right|_{z=\pm l} = 0\,. \tag{7-5.41}$$

Therefore we have the following boundary-value problem:

Determine the nontrivial solutions of the vibration equation

$$\Delta_2 Z + \frac{\partial^2 Z}{\partial z^2} + k^2 Z = 0 \tag{7-5.39'}$$

with the homogeneous boundary conditions

$$Z|_S = 0 , \tag{7-5.40'}$$

$$\left.\frac{\partial Z}{\partial z}\right|_{z=\pm l} = 0 . \tag{7-5.41'}$$

As in the case of the cylindrical hollow conductor (see p. 499), we seek the solution through the Ansatz

$$Z(M, z) = \psi(M) f(z) . \tag{7-5.42}$$

If we put this in Eq. (7-5.39'), then by consideration of the boundary condition (7-5.40), we obtain the eigenvalue problem

$$\Delta_2 \psi + \lambda \psi = 0 \text{ in } S , \tag{7-5.43}$$

$$\psi = 0 \quad \text{on } C . \tag{7-5.44}$$

This is the problem for the eigenvibrations of a membrane which is fixed along the contour C and which has the form of the perpendicular cross section S. For the determination of the function $f(z)$, we obtain by separation of variables the equation

$$f'' + (k^2 - \lambda) f = 0 \tag{7-5.45}$$

and from condition (7-5.41) the boundary conditions

$$f'(\pm l) = 0 . \tag{7-5.46}$$

It is to be noted that here, in contrast to the results for hollow conductors, k^2 is not a fixed quantity, but rather enters the equation as a parameter. Now it is necessary to find those values of k^2 for which problems (7-5.39—7-5.41) have nontrivial solutions.

Equation (7-5.45) with the conditions (7-5.46) possesses the eigenfunctions

$$f_m(z) = A_m \cos \frac{\pi m}{2l}(l - z) .$$

To these correspond the eigenvalues

$$\mu_m = \left(\frac{\pi m}{2l}\right)^2 , \quad (m = 0, 1, \cdots) ,$$

where

$$\mu_m = k_m^2 - \lambda .$$

The boundary-value problem (7-5.43—7-5.44) yields the spectrum of eigenvalues $\{\lambda_n\}$ with the corresponding system of normalized eigenfunctions $\psi_n(M)$. From this it follows that in an endovibrator only those vibrations can be present whose eigenfrequencies or resonance frequencies are equal to

$$\omega_{mn} = c\sqrt{\lambda_n + \mu_m} .$$

These frequencies correspond to the system of eigenfunctions

$$Z_{n,m}(M, z) = \bar{A}_{n,m} \psi_n(M) \cos \frac{\pi m}{2l}(l - z) \tag{7-5.47}$$

or

$$Z_{n,m}(M, z) = A_{n,m}\psi_n(M)f_m(z)\,, \qquad (7\text{-}5.47')$$

where the functions

$$f_m(z) = \sqrt{\frac{\varepsilon_m}{2l}}\cos\frac{\pi m}{2l}(l-z)\,, \quad \varepsilon_m = \begin{cases} 2 \text{ for } m \neq 0 \\ 1 \text{ for } m = 0 \end{cases}$$

are so normalized that their norms are equal to one. The solutions are determined to within the amplitude factors $A_{n,m}$. These can be obtained from the actual excitation conditions.

If the eigenfunctions $\psi_n(M)$ are known, then the field components can be calculated according to formulas (7-5.37) and (7-5.38).

If the cross section S of the endovibrator is a rectangle with sides of lengths a and b, then

$$\psi_n(M) = \psi_{p,q}(x, y) = \sqrt{\frac{4}{ab}}\sin\frac{\pi p}{a}x\sin\frac{\pi q}{b}\,, \quad (p, p = 1, 2, 3, \cdots)\,,$$

$$\lambda_n = \lambda_{p,q} = \pi^2\left(\frac{p^2}{a^2} + \frac{q^2}{b^2}\right),$$

$$Z_{n,m} = A_{m,p,q}\sqrt{\frac{2\varepsilon_m}{abl}}\sin\frac{\pi p}{a}x\sin\frac{\pi q}{b}y\cos\frac{\pi m}{2l}(l-z)\,.$$

In this case, the smallest eigenfrequency is equal to

$$\omega_{0,1,1} = c\sqrt{\lambda_{1,1}} = c\pi\sqrt{\frac{1}{a^2} + \frac{1}{b^2}}\,.$$

It corresponds to the maximum wave length attainable:

$$\Lambda_0 = \frac{2}{\sqrt{\frac{1}{a^2} + \frac{1}{b^2}}}\,.$$

In particular, for $a = b$ the maximum wave length is equal to

$$\Lambda_0 = a\sqrt{2}\,.$$

It is equal, therefore, to the diagonal of a square which is formed by the perpendicular section. Consequently, in such an endovibrator, only eigenvibrations with frequencies

$$\omega \geqq \omega_{0,1,1}$$

or wave lengths

$$\Lambda \leqq \Lambda_0$$

are possible. Completely analagous results are obtained for magnetic type eigenvibrations ($E_z = 0$). In this case, we set

$$\boldsymbol{E} = ik\,\mathrm{rot}\,\hat{\mathfrak{Z}}\,,$$

$$\boldsymbol{H} = \mathrm{grad}\,\mathrm{div}\,\hat{\mathfrak{Z}} + k^2\hat{\mathfrak{Z}}$$

with

$$\hat{\mathfrak{Z}} = \hat{Z}\boldsymbol{i}_z\,.$$

For the determination of $\hat{Z}(M, z)$ we have equation (7-5.39) with the boundary conditions

$$\frac{\partial \hat{Z}}{\partial \nu}\Big|_{\Sigma} = 0 , \tag{7-5.40'}$$

$$\hat{Z}|_{z=\pm l} = 0 . \tag{7-5.41'}$$

The solutions are

$$\hat{Z}_{n,m} = \hat{A}_{n,m}\psi_m(M) \sin \frac{\pi m}{l}(l - z) . \tag{7-5.48}$$

Here $\hat{\psi}_n(M)$ is understood to be the eigenfunctions of the membrane S with the boundary condition $\partial\hat{\psi}/\partial\nu = 0$ on C.

2. *Electromagnetic energy of eigenvibrations.* We shall calculate the energy of the electric and magnetic fields of a standing wave in a cylindrical endovibrator.

For simplicity, we shall limit ourselves to waves of the electrical type. If we consider that the periodic dependence of $\boldsymbol{E}$ and $\boldsymbol{H}$ are given by the factor $e^{-i\omega t}$, then from formulas (7-5.37) and (7-5.38), provided we take only the real part, we obtain

$$\begin{aligned} E_x &= \frac{\partial^2 Z}{\partial z \partial x} \cos \omega t , \\ E_y &= \frac{\partial^2 Z}{\partial z \partial y} \cos \omega t , \\ E_z &= \left(\frac{\partial^2 Z}{\partial z^2} + k^2 Z\right) \cos \omega t , \end{aligned} \tag{7-5.49}$$

and

$$\begin{aligned} H_x &= -k\frac{\partial Z}{\partial y} \sin \omega t , \\ H_y &= k\frac{\partial Z}{\partial x} \sin \omega t , \\ H_z &= 0 . \end{aligned} \tag{7-5.50}$$

For calculation of the energy of the electric and magnetic fields, the known formulas

$$E_{\mathrm{El}}(t) = \frac{c}{8\pi}\iiint_T \boldsymbol{E}^2 \, d\tau , \tag{7-5.51}$$

$$E_{\mathrm{M}}(t) = \frac{c}{8\pi}\iiint_T \boldsymbol{H}^2 \, d\tau \tag{7-5.52}$$

are used, where the integration is to be extended over the spatial region T of the endovibrator.

If expression (7-5.49) is introduced into (7-5.51), then by using formula (7-5.47') we obtain[39]

[39] The indices m, n can be temporarily dropped.

$$E_{\mathrm{El}}(t) = \frac{A^2 c}{8\pi}\cos^2 \omega t \left\{ \int_S \int \left[\left(\frac{\partial \phi}{\partial x}\right)^2 + \left(\frac{\partial \phi}{\partial y}\right)^2 \right] d\sigma \int_{-l}^{l} [f'(z)]^2 \, dz + \int_S \int \phi^2 \, d\sigma \int_{-l}^{l} (f'' + k^2 f)^2 \, dz \right\} .$$

After several simple calculations, we find

$$\int_{-l}^{l} [f'(z)]^2 \, dz = ff' \Big|_{-l}^{l} - \int_{-l}^{l} ff'' \, dz = (k^2 - \lambda) \int_{-l}^{l} f^2 \, dz = k^2 - \lambda , \qquad (7\text{-}5\text{-}52')$$

$$\int_{-l}^{l} (f'' + k^2 f)^2 \, dz = \lambda^2 \int_{-l}^{l} f^2 \, dz = \lambda^2 , \qquad (7\text{-}5.53)$$

since f is normalized by

$$\int_{-l}^{l} f^2 \, dz = 1 . \qquad (7\text{-}5.54)$$

For the calculation of the integral extended over S, we use the first Green's formula, the equation for the functions ϕ_n, the boundary conditions, and the normalizing condition

$$\int_S \int \left[\left(\frac{\partial \phi}{\partial x}\right)^2 + \left(\frac{\partial \phi}{\partial y}\right)^2 \right] dx \, dy = \int_S \int (\nabla_2 \phi)^2 \, d\sigma = -\int_S \int \phi \Delta_2 \phi \, d\sigma + \int_C \phi \frac{\partial \phi}{\partial \nu} \, ds = \lambda \int_S \int \phi^2 \, d\sigma = \lambda , \qquad (7\text{-}5.55)$$

where ∇_2 is the del operator in the S plane and Δ_2 is the two-dimensional Laplace operator. As a result, for the energy of the electrical field we obtain the expression

$$E_{\mathrm{El}}(t) = \frac{A^2 c}{8\pi} k^2 \lambda \cos^2 \omega t . \qquad (7\text{-}5.56)$$

For the energy of the magnetic fields on the basis of formulas (7-5.50), (7-5.52), (7-5.47'), we have

$$E_{\mathrm{M}}(t) = \frac{A^2 c k^2}{8\pi} \int_S \int \left[\left(\frac{\partial \phi}{\partial x}\right)^2 + \left(\frac{\partial \phi}{\partial y}\right)^2 \right] dx \, dy \int_{-l}^{l} f^2 \, dz \sin^2 \omega t ,$$

from which we obtain

$$E_{\mathrm{M}}(t) = \frac{A^2 c k^2}{8\pi} \lambda \sin^2 \omega t \qquad (7\text{-}5.57)$$

by consideration of (7-5.54) and (7-5.55). The total energy of the electromagnetic field is independent of time:

$$E = E_{\mathrm{El}}(t) + E_{\mathrm{M}}(t) = \frac{A^2 c k^2}{8\pi} \lambda . \qquad (7\text{-}5.58)$$

Formulas (7-5.56) and (7-5.57) show that in a standing wave there occurs an interchangeable transformation of electrical energy to magnetic energy, and conversely. Here the time average value during a period for the energy of the electric field

$$\bar{E}_{\mathrm{El}} = \tfrac{1}{2}\frac{A^2ck^2}{8\pi}\lambda = \tfrac{1}{2}E \tag{7-5.59}$$

equals the corresponding average value of the energy of a magnetic field

$$\bar{E}_{\mathrm{M}} = \tfrac{1}{2}\frac{A^2ck^2}{8\pi}\lambda = \tfrac{1}{2}E\,. \tag{7-5.60}$$

3. *Generation of vibrations in an endovibrator.* For the generation of a field in an endovibrator by external sources, a connecting element must be introduced through a crevice in the wall of the endovibrator. A coil or a wire can be used as such a connecting element, which then acts as a small antenna. So that the field in the endovibrator may not be perturbed by the connecting element, the dimensions of the connecting element must be very much smaller than the wave length. There is still another series of possibilities of exciting vibrations in the endovibrator, for example, through an electronic sheaf which penetrates the endovibrator (through an opening in its wall).

In order to solve the problem of excitation of an endovibrator through an antenna placed in its interior or, in an extreme case, through an elementary dipole, the finite conductivity of the walls of the endovibrator must be considered. Otherwise, it would be impossible to cause vibrations in the endovibrator. The finite conductivity of the walls can be considered by means of the Leontovitch condition.

Here we shall treat the excitation of a spherical endovibrator by a dipole. In this case, a simple analytic solution can be obtained.[40] At the center of of a sphere of radius r_0, we place a dipole. This vibrates with the frequency ω and the amplitude 1 and has the direction of the z-axis. We seek the field within the sphere by taking into consideration the finite conductivity of the walls of the sphere.

In this case, the field strengths $\boldsymbol{E}$ and $\boldsymbol{H}$ can be expressed by a single function U as follows:

$$\begin{aligned} E_r &= \frac{i}{\rho\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial U}{\partial\theta}\right), \\ E_\theta &= -\frac{i}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial U}{\partial\theta}\right), \\ H_\varphi &= \frac{\partial U}{\partial\theta}\,. \end{aligned} \tag{7-5.61}$$

The remaining components E_φ, H_r, H_θ are equal to zero.

Since the dipole has the direction of the z-axis ($\theta = 0$), the field obviously cannot be dependent on φ. The function U satisfies the equation

$$\frac{1}{\rho^2}\frac{\partial}{\partial\rho}\left(\rho^2\frac{\partial U}{\partial\rho}\right) + \frac{1}{\rho^2\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial U}{\partial\theta}\Big) + U = 0\,, \tag{7-5.62}$$

where $\rho = kr$. For $\rho \to 0$, U has a singularity of the form

[40] See S. M. Rytov, *Doklady SSSR* 51, 2, 1946.

$$\frac{ie^{ikr}}{r^2} = \frac{ik^2 e^{i\rho}}{\rho^2} . \tag{7-5.63}$$

On the surface of the sphere ($\rho = \rho_0$) the Leontovitch condition

$$E_\theta = aH_\varphi \tag{7-5.64}$$

must be satisfied. Here

$$a = \mu kd \sqrt{\frac{i}{2}}, \quad \left(d = \frac{c}{\sqrt{2\pi\mu\sigma\omega}}\right) \tag{7-5.65}$$

is the effective thickness of the skin layer.

From expressions (7-5.64) and (7-5.61) we can obtain

$$\left[\frac{\partial}{\partial\rho}(\rho U) - i\rho_0 a U\right]_{\rho=\rho_0} = 0$$

or

$$\rho_0 \frac{\partial U}{\partial\rho}\bigg|_{\rho=\rho_0} + (1 - i\rho_0 a)U\bigg|_{\rho=\rho_0} = 0 \tag{7-5.66}$$

as the boundary condition for the function U.

A solution of Eq. (7-5.62) with singularity (7-5.63) is obviously given by the function

$$U = -k^2 \sqrt{\frac{\pi}{2\rho}} [H^{(1)}_{3/2}(\rho) + CJ_{3/2}(\rho)] P_1(\cos\theta) .$$

Here $P_1(\cos\theta)$ is the Legendre polynomial of first order, $H^{(1)}_{3/2}$ is the Hankel function of the first kind, and $J_{3/2}$ is the Bessel function:

$$P_1(\cos\theta) = \cos\theta ,$$

$$H^{(1)}_{3/2}(\rho) = \sqrt{\frac{2}{\pi\rho}} e^{i\rho}\left(\frac{1}{i\rho} - 1\right) ,$$

$$J_{3/2}(\rho) = \sqrt{\frac{2}{\pi\rho}}\left(\frac{\sin\rho}{\rho} - \cos\rho\right) .$$

The constant C can be calculated from the boundary condition (7-5.66):

$$C = -e^{i\rho_0} \frac{1 - \frac{1}{\rho_0^2} - \frac{i}{\rho_0} + a\left(\frac{1}{i\rho_0} - 1\right)}{i\left[\frac{\cos\rho_0}{\rho_0} + \left(1 - \frac{1}{\rho_0^2}\right)\sin\rho_0 - ia\left(\frac{\sin\rho_0}{\rho_0} - \cos\rho_0\right)\right]} .$$

This solution can be used for the determination of the magnitude of the loss in the walls of the sphere. The amount absorbed by the walls

$$Q = \frac{\mu\omega d}{16\pi} \int_0^\pi |H_\varphi|^2 2\pi\rho_0^2 \sin\theta \, d\theta$$

can be calculated directly and is equal to

$$Q = \frac{\mu\omega k^4 d}{6} \frac{1}{|B - iaA|^2} ,$$

where

$$A = \frac{\sin \rho_0}{\rho_0} - \cos \rho_0 , \quad B = \frac{\cos \rho_0}{\rho_0} + \left(1 - \frac{1}{\rho_0^2}\right) \sin \rho_0 .$$

If the dipole is not at the center of the sphere, the calculation of the field is essentially very difficult; however, the solution can be given as an infinite series.

3. Skin effect

In contrast to direct current, alternating current is not distributed uniformly in the cross section of the conductor, but shows a greater density on the surface of the conductor. This phenomena is known as the skin effect.[41]

For simplicity, we consider an infinitely long homogeneous cylindrical conductor ($\mu = \text{const}$, $\sigma = \text{const}$) through which flows an alternating current. We shall assume that the total current $I = I_0 e^{i\omega t}$ flowing through the conductor is known.

If we neglect the displacement current[42] and the process is stationary, that is, the time dependence follows the law $e^{i\omega t}$, then we obtain Maxwell's equations in the form

$$\text{rot } \boldsymbol{H} = \frac{4\pi\sigma}{c} \boldsymbol{E} , \tag{7-5.67}$$

$$\text{rot } \boldsymbol{E} = - ik\mu \boldsymbol{H} , \tag{7-5.68}$$

$$\text{div } \boldsymbol{E} = 0 , \tag{7-5.69}$$

$$\text{div } \boldsymbol{H} = 0 \tag{7-5.70}$$

with $k = \omega/c$ after leaving out the factors $e^{i\omega t}$.

In the present case, Eq. (7-5.69) and (7-5.70) obviously follow from Eq. (7-5.67) and (7-5.68).

Now we introduce cylindrical coordinates r, ϑ, φ such that the z-axis coincides with the axis of the conductor. Then, because of axial symmetry, all of the quantities occurring depend only on r.

Since in the present case the vector $\boldsymbol{E}$ has the direction of the z-axis, it follows from Eqs. (7-5.67) and (7-5.68):

$$\frac{1}{r} \frac{d}{dr}(rH_\varphi) = \frac{4\pi\sigma}{c} E_z , \tag{7-5.67'}$$

$$\frac{d}{dr} E_z = ik\mu H_\varphi . \tag{7-5.68'}$$

[41] See also I. E. Tamm, *Basic Theory of Electricity*, Moscow, 1946. The German book was prepared by K. Simonyi, *Theoretical Electrotechnology*, Berlin, 1956.

[42] Within the conductors, particularly inside of the metals, the density of the displacement current is very small in comparison with the conduction current density: $j_v \ll j = \sigma E$. In our case, this condition is equivalent to the requirement

$$\varepsilon\omega \ll \sigma .$$

Since for solid metals $\sigma \approx 10^{17}$ absolute units, for all usable frequencies in technology the displacement current can be neglected.

By elimination of H_φ, we obtain

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{dE_z}{dr}\right) = i\frac{4\pi\sigma\mu k}{c}E_z \,. \tag{7-5.71}$$

To determine the boundary condition which must hold on the conductor surface for $r = R$, we use the hypothesis that the total current I_0 flowing through the cylinder is known.

Therefore, we write the first Maxwell equation (7-5.67) in the integral form

$$\oint_C H_s\, ds = \frac{4\pi}{c} I_0 \,.$$

Here C is a curve enclosing the conductor and H_s is the tangential component of the vector $\boldsymbol{H}$ on C. If the curve is chosen as a circle of radius $r = R$ on the surface of the conductor, then we obtain

$$\int_0^{2\pi} H_\varphi(R)\, d\varphi = \frac{4\pi}{cR} I_0$$

or

$$H_\varphi(R) = \frac{2I_0}{cR} \,, \tag{7-5.72}$$

From this, together with (7-5.68), we find

$$\left.\frac{dE_z}{dr}\right|_{r=R} = \frac{2ik\mu}{cR} I_0 \,. \tag{7-5.73}$$

Therefore, the Bessel differential equation

$$E_z''(r) + \frac{1}{r}E_r'(r) + (\alpha\sqrt{-i})^2 E_z(r) = 0 \,, \quad \left(\alpha^2 = \frac{4\pi\sigma\mu\omega}{c^2}\right) \tag{7-5.71'}$$

is to be solved with the boundary condition

$$E_z'(R) = \frac{2ik\mu}{cR} I_0 \,. \tag{7-5.73}$$

Moreover, the solution for $r = 0$ is to be bounded, that is, the requirement

$$E_z(0) < \infty \tag{7-5.74}$$

for $r = 0$ is to be satisfied. The general solution of the Bessel differential equation (7-5.71′) has the form

$$AJ_0(\alpha r\sqrt{-i}) + BN_0(\alpha r\sqrt{-i}) \,, \tag{7-5.75}$$

where J_0 and N_0 are the Bessel functions of the first and second kinds (see Appendix, Section A 2), while A and B are constants still to be determined.

The function N_0, for $r = 0$, has a logarithmic singularity. Therefore, because of the requirement (7-5.74), $B = 0$ must be chosen. Consequently,

$$E_z(r) = AJ_0(\alpha r\sqrt{-i}) \,. \tag{7-5.76}$$

The coefficient A is determined from the boundary condition (7-5.73):

$$A = \frac{2\sqrt{-ik\mu I_0}}{\alpha c R J_1(\alpha R\sqrt{-i})} . \tag{7-5.77}$$

From this, for the current density

$$j = \sigma E_z ,$$

we obtain the expression

$$j(r) = \frac{I_0 \alpha \sqrt{-i}}{2\pi R J_1(\alpha R\sqrt{-i})} J_0(\alpha r\sqrt{-i}) . \tag{7-5.78}$$

On the right side of (7-5.78) stand the Bessel functions with complex arguments

$$x\sqrt{-i} = \frac{1-i}{\sqrt{2}}x .$$

For these functions we ordinarily use the expressions

$$J_0(x\sqrt{-i}) = \mathrm{ber}_0 x + i\,\mathrm{bei}_0 x ,$$
$$J_1(x\sqrt{-i}) = \mathrm{ber}_1 x + i\,\mathrm{bei}_1 x .$$

The expression for the functions ber x and bei x can easily be given if we start with the series expansions of the Bessel functions. For example, we have

$$J_0(x\sqrt{-i}) = J_0(xi\sqrt{i}) = 1 - \frac{\left(\frac{x}{2}\right)^2(-1)i}{(1!)^2} + \frac{\left(\frac{x}{2}\right)^4(-1)}{(2!)^2} - \frac{\left(\frac{x}{2}\right)^6 i}{(3!)^2} + \frac{\left(\frac{x}{2}\right)^8}{(4!)^2} - \cdots$$
$$= \left\{1 - \frac{\left(\frac{x}{2}\right)^4}{(2!)^2} + \frac{\left(\frac{x}{2}\right)^8}{(4!)^2} - \cdots\right\} + i\left\{\frac{\left(\frac{x}{2}\right)^2}{(1!)^2} - \frac{\left(\frac{x}{2}\right)^6}{(3!)^2} + \cdots\right\} .$$

From this we obtain

$$\mathrm{ber}_0 x = 1 - \frac{\left(\frac{x}{2}\right)^4}{(2!)^2} + \frac{\left(\frac{x}{2}\right)^8}{(4!)^2} - \cdots , \tag{7-5.79}$$

$$\mathrm{bei}_0 x = \frac{\left(\frac{x}{2}\right)^2}{(1!)^2} - \frac{\left(\frac{x}{2}\right)^6}{(3!)^2} + \cdots . \tag{7-5.80}$$

In a similar manner, it is easily seen that

$$\mathrm{ber}_1 x = \frac{1}{\sqrt{2}}\left\{\frac{x}{2} + \frac{\left(\frac{x}{2}\right)^3}{1!2!} - \frac{\left(\frac{x}{2}\right)^5}{2!3!} - \frac{\left(\frac{x}{2}\right)^7}{3!4!} + \cdots\right\} , \tag{7-5.81}$$

$$\mathrm{bei}_1 x = \frac{1}{\sqrt{2}}\left\{-\frac{x}{2} + \frac{\left(\frac{x}{2}\right)^3}{1!2!} + \frac{\left(\frac{x}{2}\right)^5}{2!3!} - \frac{\left(\frac{x}{2}\right)^7}{3!4!} - \cdots\right\} . \tag{7-5.82}$$

In applications, the derivatives

$$\mathrm{ber}_0' x , \quad \mathrm{bei}_0' x$$

also occur. For these we find

$$J_1(x\sqrt{-i}) = \sqrt{-i}\,(\mathrm{bei}_0' x - i\,\mathrm{ber}_0' x)\,. \tag{7-5.83}$$

With the use of these functions, formula (7-5.78) for the current density can be written in the following manner:

$$j(r) = \frac{I_0 \alpha}{2\pi R}\,\frac{\mathrm{ber}_0 \alpha r + i\,\mathrm{bei}_0 \alpha r}{\mathrm{bei}_0' \alpha R \quad i\,\mathrm{ber}_0' \alpha R}$$

or

$$j(r) = \frac{I_0 \alpha}{2\pi R}\left\{\frac{(\mathrm{ber}_0 \alpha r\,\mathrm{bei}_0' \alpha R \quad \mathrm{bei}_0 \alpha r\,\mathrm{ber}_0' \alpha R)}{(\mathrm{bei}_0' \alpha R)^2 + (\mathrm{ber}_0' \alpha R)^2}\right.$$
$$\left. + i\,\frac{(\mathrm{bei}_0 \alpha r\,\mathrm{bei}_0' \alpha R + \mathrm{ber}_0 \alpha r\,\mathrm{ber}_0' \alpha R)}{(\mathrm{bei}_0' \alpha R)^2 + (\mathrm{ber}_0' \alpha R)^2}\right\}. \tag{7-5.84}$$

For the absolute value of these expressions, we obtain

$$|j(r)| = \frac{I_0 \alpha}{2\pi R}\,\sqrt{\frac{(\mathrm{ber}_0 \alpha r)^2 + (\mathrm{bei}_0 \alpha r)^2}{(\mathrm{ber}_0' \alpha R)^2 + (\mathrm{bei}_0' \alpha R)^2}}\,. \tag{7-5.85}$$

The ratio

$$\frac{|j(r)|}{|j(R)|} = \sqrt{\frac{(\mathrm{ber}_0 \alpha r)^2 + (\mathrm{bei}_0 \alpha r)^2}{(\mathrm{ber}_0 \alpha R)^2 + (\mathrm{bei}_0 \alpha R)^2}} \tag{7-5.86}$$

represents a characteristic quantity of the distribution of current in a cross section. We shall calculate the current distribution in a cross section for two frequencies, $\omega_1 = 314$ (50 cycles in a second) and $\omega_2 = 314 \cdot 10^4$ ($5 \cdot 10^5$ cycles in a second). For the calculation, the cgs system is appropriate.

The reasonings above were carried out entirely in the Gaussian system of measure. Therefore, when transforming to the cgs system, we should bear in mind that $\mu_{\mathrm{cgs}} = \mu_{\mathrm{Gauss}} \cdot 1/c^2$. All the remaining quantities which occur in formulas (7-5.78), (7-5.84), (7-5.85), and (7 5.86) coincide in both systems. Thus, in the cgs system we have

$$\alpha^2 = 4\pi\mu\sigma\omega\,.$$

Now let $\sigma = 57 \cdot 10^5$ cgs units. Then $\alpha_1 = 0.4444$ (for ω_1) and $\alpha_2 = 44.44$ (for ω_2). We calculate the ratio of the absolute value of the current density (7-5.86) in the case of the lower frequency $\omega_1 = 314$ for both values, $r = 0$ and $r = 0.5R$. Here we set $R = 1$.

By considering the values[43]

$$\mathrm{ber}_0 0 \quad = 1\,,$$
$$\mathrm{bei}_0 0 \quad - 0\,,$$
$$\mathrm{ber}_0 0.222 - 1 - 0.000036 + \cdots \quad 0.999964\,,$$
$$\mathrm{bei}_0 0.222 - 0.0123 \quad 0.000002 + \cdots - 0.0123000\,,$$
$$\mathrm{ber}_0 0.444 - 1 \quad 0.00061 + \cdots \quad 0.9939\,,$$
$$\mathrm{bei}_0 0.444 \quad 0.493 - 0.0003 + \cdots - 0.4990\,,$$

[43] See E. Jahnke and F. Emde, *Tables of Special Functions*, Dover, 1965.

we obtain

$$\left.\frac{j(0)}{j(R)}\right|_{R=1} = 0.9994\,, \quad \left.\frac{j(0.5R)}{j(R)}\right|_{R=1} = 0.9999\,.$$

that is, for smaller frequencies, the current is distributed almost uniformly in the cross section (the skin effect is absent).

Now we consider the second case, $\omega_2 = 314 \cdot 10^4$. Since α is large, for the purpose of proof we do not use the series expansion of the function ber x and bei x, but rather the asymptotic representations

$$J_0(\alpha r\sqrt{-i}) = \frac{1}{\sqrt{2\pi\alpha r}} e^{(\alpha r/\sqrt{2})\, i((\alpha r/\sqrt{2})\, -\, (\pi/8))}\,,$$

$$J_1(\alpha R\sqrt{-i}) = \frac{1}{\sqrt{2\pi\alpha R}} e^{(\alpha R/\sqrt{2})\, i((\alpha R/\sqrt{2})-(\pi/8))}\,.$$

From this we obtain for $r = 0.9R$ and $R = 1$

$$\left.\frac{j(0.9R)}{j(R)}\right|_{R=1} = \sqrt{2}\, e^{-(44\cdot 0.1/\sqrt{2})} \approx 0.28\,.$$

These results show a very rapid decrease of the current density with increasing penetration depth for higher frequencies. Finally, we should note that the skin effect in practice is often used in the hardening of metals.

4. The propagation of radio waves on the surface of the earth

Problems which are related to the propagation of radio waves in free space or in spaces with surfaces of separation, have vast theoretical and practical significance. A large quantity of work has been devoted to these questions.

In the following discussion we shall investigate the influence of the earth on the propagation of radio waves which are radiated by a vertical dipole. We shall consider the earth's surface to be a plane.[44]

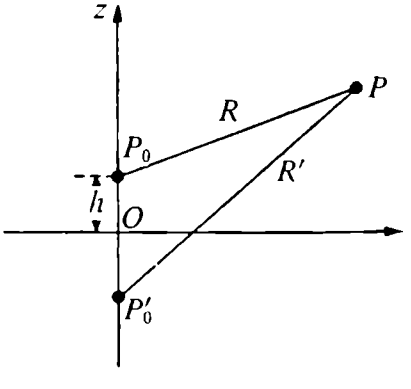


FIG. 91

[44] This problem was solved for the first time by A. Sommerfeld in 1909. The original solution of Sommerfeld contained an error which was corrected by V. A. Fock.

Above the surface of the earth at a distance h at the point P_0, we place a dipole which sends out periodic vibrations with frequency ω. The surface of the earth is considered to be a plane coinciding with the plane $z = 0$, and the z-axis with the axis of the dipole (Figure 91). Now we assume that in the atmosphere ($z > 0$), $\varepsilon_0 = \mu_0 = 1$ and $\sigma_0 = 0$. Furthermore, we assume that the earth ($z < 0$) has a dielectric constant ε and a conductivity σ and that the magnetic permeability μ is equal to one; ε and σ are taken to be constant.

Then our problem is to determine the strength of the electromagnetic field generated by the dipole. The propagation of electromagnetic waves is described by the Maxwell's equations.

As was shown in Section 5-4.2 the solution of Maxwell's equations reduces to the solution of the vibration equation for the polarization potential $\mathfrak{Z}$,[45]

$$\Delta\mathfrak{Z} + k^2\mathfrak{Z} = 0 \tag{7-5.87}$$

with

$$k^2 = \begin{cases} k_0^2 = \dfrac{\omega^2}{c^2}, & z > 0, \\ k_3^2 = \dfrac{\varepsilon\omega^2 + i\sigma\omega}{c^2}, & z < 0. \end{cases}$$

The relation of the potential $\mathfrak{Z}$ to the field strengths $\boldsymbol{E}$ and $\boldsymbol{H}$ is given by the expressions

$$\begin{aligned} \boldsymbol{E} &= k^2\mathfrak{Z} + \operatorname{grad}\operatorname{div}\mathfrak{Z}, \\ \boldsymbol{H} &= -i\frac{k^2}{k_0}\operatorname{rot}\mathfrak{Z}. \end{aligned} \tag{7-5.88}$$

In our case, the vector $\mathfrak{Z}$ is directed parallel to the radiating dipole:

$$\mathfrak{Z} = (0, 0, Z_z), \quad Z_z = Z_z(r, z). \tag{7-5.89}$$

Then if we set

$$n^2 = \varepsilon + i\frac{\sigma}{\omega},$$

we obtain

$$k_3^2 = n^2 k_0^2.$$

Expressions (7-5.88) and (7-5.89) give

$$E_r = \frac{\partial}{\partial r}\frac{\partial Z_0}{\partial z}, \quad H_\varphi = -ik_0^2\frac{\partial Z_0}{\partial r}, \quad E_\varphi = H_r = 0 \quad \text{for} \quad z > 0, \tag{7-5.90}$$

$$E_r = \frac{\partial}{\partial r}\frac{\partial Z_3}{\partial z}, \quad H_\varphi = -\frac{ik_3^2}{k_0}\frac{\partial Z_3}{\partial r}, \quad E_\varphi = H_r = 0 \quad \text{for} \quad z < 0. \tag{7-5.91}$$

To establish the boundary conditions for $z = 0$, we start, therefore, from the fact that the tangential components of the field strength are continuous. This is the case if, as a result of (7-5.90) and (7-5.91), we set

[45] Since the process here is considered an induced process, the time factor $e^{-i\omega t}$ can be dropped.

$$\frac{\partial Z_0}{\partial z} = \frac{\partial Z_3}{\partial z}, \quad Z_0 = n^2 Z_3 \quad \text{for} \quad z = 0 . \tag{7-5.92}$$

We seek the solution of Eq. (7-5.87) under the boundary conditions (7-5.92) as a superposition of particular solutions of the form

$$J_0(\lambda r)e^{\pm \mu z}, \quad (k^2 = \lambda^2 + \mu^2) .$$

In the case of an unbounded region we obtain a continuous spectrum of eigenvalues λ. For this reason, the solution Z can be obtained in the form

$$Z = \int_0^\infty F(\lambda) J_0(\lambda r) e^{\pm \mu z} \, d\lambda . \tag{7-5.93}$$

The sign of μ is so chosen that the integral (7-5.93) converges. The next function $F(\lambda)$, still to be determined, represents the amplitude factors of the individual vibrations.

Now we use the integral representation of the potentials (see Appendix, Section A-2.5)

$$\begin{gathered} Z_0 = \frac{e^{ikR}}{R} = \int_0^\infty J_0(\lambda r) e^{-\mu |z|} \frac{\lambda \, d\lambda}{\mu} \\ (\mu = \sqrt{\lambda^2 - k^2}, \quad R = \sqrt{r^2 + z^2}) . \end{gathered} \tag{7-5.94}$$

For this we consider two different regions:

(a) Air ($z > 0$). The field here has the form

$$Z_0 = Z_{\text{prim}} + Z_{\text{sec}} , \tag{7-5.95}$$

where

$$Z_{\text{prim}} = \frac{e^{ikR}}{R}$$

is the field potential of the primary excitation generated by the dipole itself and Z_{sec} is the field potential of the secondary excitation which is produced by the current originating in the earth.

By the use of (7-5.93), (7-5.94), and (7-5.95) we can write

$$\begin{aligned} Z_{\text{prim}} &= \int_0^\infty J_0(\lambda r) e^{-\mu |z-h|} \frac{\lambda \, d\lambda}{\mu} , \\ Z_{\text{sec}} &= \int_0^\infty F(\lambda) J_0(\lambda r) e^{-\mu (z+h)} \, d\lambda . \end{aligned} \tag{7-5.96}$$

Here $F(\lambda)$ is still undetermined.

(b) Earth ($z < 0$). Here only the secondary excitation appears. For this we can write

$$Z_3 = \int_0^\infty F_3(\lambda) J_0(\lambda r) e^{\mu_3 z - \mu h} \, d\lambda , \tag{7-5.97}$$

where $\mu_3^2 = k_3^2 - \lambda^2$. Since $z < 0$, the signs of the exponents guarantee the convergence of the integral.

For the determination of the functions $F(\lambda)$ and $F_3(\lambda)$ we use conditions (7-5.91) which yield

$$\int_0^\infty J_0(\lambda r)e^{-\mu h}[\lambda - \mu F(\lambda) - \mu_3 F_3(\lambda)]\,d\lambda = 0\,,$$
$$\int_0^\infty J_0(\lambda r)e^{-\mu h}[\lambda + \mu F(\lambda) - n^2\mu F_3(\lambda)]\,d\lambda = 0\,. \tag{7-5.98}$$

Conditions (7-5.98) are satisfied if we set

$$\mu F(\lambda) + \mu_3 F_3(\lambda) = \lambda\,,$$
$$\mu F(\lambda) - n^2\mu F_3(\lambda) = -\lambda\,. \tag{7-5.99}$$

If we solve these equations, then for $F(\lambda)$ and $F_3(\lambda)$ we obtain

$$F(\lambda) = \frac{\lambda}{\mu}\left(1 - \frac{2\mu_3}{n^2\mu + \mu_3}\right),$$
$$F_3(\lambda) = \frac{2\lambda}{n^2\mu = \mu_3}\,. \tag{7-5.100}$$

If we put expression (7-5.100) into formulas (7-5.96) and (7-5.97), then we arrive at the following expressions for the polarization potential of the fields of a vertical dipole:

$$Z_0 = \int_0^\infty J_0(\lambda r)e^{-\mu|z-h|}\frac{\lambda\,d\lambda}{u} + \int_0^\infty J_0(\lambda r)e^{-\mu|z+h|}\frac{\lambda\,d\lambda}{\mu}$$
$$- 2\int_0^\infty J_0(\lambda r)e^{-\mu|z+h|}\frac{\mu_3}{n^2\mu + \lambda_3}\frac{\lambda\,d\lambda}{\mu}\,, \tag{7-5.101}$$
$$Z_3 = 2\int_0^\infty J_0(\lambda r)e^{\mu_3 z - \mu h}\frac{\lambda\,d\lambda}{n^2\mu + \mu_3}\,.$$

We denote by $R = \sqrt{r^2 + (z-h)^2}$ the distance from the test point to the dipole and by $R' = \sqrt{r^2 + (z+h)^2}$ the distance of the test point from the mirror image of the dipole and the plane $z = 0$. By using the representation (7-5.94), we can write

$$Z_0 = \frac{e^{ikR}}{R} + \frac{e^{ikR'}}{R'} - 2\int_0^\infty J_0(\lambda r)e^{-\mu(z+h)}\frac{\mu_3}{n^2\mu + \mu_3}\frac{\lambda\,d\lambda}{\mu}\,. \tag{7-5.101'}$$

Now we consider a few limiting cases.

1. *The perfectly conducting earth.* In this case $\sigma \to \infty$; consequently, $|k_3|$ and $|n|$ tend towards infinitely. From formulas (7-5.101) and (7-5.101') it follows that

$$Z_0 = \frac{e^{ikR}}{R} + \frac{e^{ikR'}}{R'}\,, \quad Z_3 = 0\,.$$

These results can be obtained directly if we solve the problem by the method of images.

2. *The dipole in a homogeneous medium.* In this case, $k_0 = k_3$, $n = 1$, and $\mu = \mu_3$. Formulas (7-5.101) and (7-5.101'), therefore, give

$$Z_0 = Z_3 = \int_0^\infty J_0(\lambda r)e^{-\mu|z-h|}\frac{\lambda\,d\lambda}{\mu} - \frac{e^{ikR}}{R}\,,$$

that is, only a primary excitation exists, which was to be expected.

The integral representations of (7-5.101) are too complicated for further investigations and for practical applications. The integrands have branch points and poles. Sommerfeld investigated these integrals with the help of approximation methods by deforming the paths of integration. He obtained the following approximation formula for the field in the neighborhood of the earth's surface:

$$Z_0 = 2\frac{e^{ik_0 r}}{r}\left(1 + i\sqrt{\pi\rho}\, e^{-\rho} - 2\sqrt{\rho}\, e^{-\rho}\int_0^{\sqrt{\rho}} e^{\alpha^2}\, d\alpha\right), \qquad (7\text{-}5.102)$$

where the quantity ρ (the so-called "numerical distance") is related to the pole p of the integrand in (7-5.101) by the expression

$$\rho = i(k_0 - p)r\,.$$

Formula (7-5.102) coincides with the formulas obtained by other authors (Weil, Van der Pol, Fock) using entirely different methods.

APPENDIX: SPECIAL FUNCTIONS

A-1. INTRODUCTION

1. Differential equations of special functions

The method of separation of variables, as we have seen from the solution of simple problems in Chapters 2 and 3, leads to the eigenvalue problem for the homogeneous equation

$$\boldsymbol{L}[y] + \lambda\rho y(x) = 0 , \qquad (a < x < b) , \tag{A-1.1}$$

where

$$\boldsymbol{L}[y] = \frac{d}{dx}\left[k(x)\frac{dy}{dx}\right] - q(x)y , \qquad (k(x) > 0) .$$

Because of the equation satisfied by the trigonometric functions

$$y'' + \lambda y = 0 , \qquad y(0) = y(l) = 0 , \tag{A-1.2}$$

which corresponds to the case $a = 0, b = l, q = 0, k = \rho = \text{const}$, complicated boundary-value problems also occur. Thus, for example, the problem of eigenvibrations of a circular membrane (of radius r_0) leads to the Bessel differential equation

$$\frac{1}{x}\frac{d}{dx}\left(x\frac{dy}{dx}\right) + \left(\lambda - \frac{n^2}{x^2}\right)y = 0 \tag{A-1.3}$$

or

$$(xy')' - \frac{n^2}{x}y + \lambda xy = 0 ,$$

$$\left[k(x) = x , \quad q(x) = \frac{n^2}{x} , \quad \rho(x) = x , \quad a = 0 , \quad b = r_0\right] ;$$

the problem of eigenvibrations of a sphere leads to the Legendre differential equation

$$\begin{gathered}[(1 - x^2)y']' + \lambda y = 0 \\ [k(x) = 1 - x^2 , \quad q = 0 , \quad \rho = 1 , \quad a = -1 , \quad b = 1]\end{gathered} \tag{A-1.4}$$

and to the equation of the associated functions

$$\begin{gathered}[(1 - x^2)y']' - \frac{n^2}{1 - x^2}y + \lambda y = 0 \\ \left[k(x) = 1 - x^2 , \quad \rho = 1 , \quad q = \frac{n^2}{1 - x^2} , \quad a = -1 , \quad b = 1\right] .\end{gathered} \tag{A-1.5}$$

The characteristic peculiarity of these equations consists of the fact that the coefficient $k(x)$, at least at one point of the interval $[a, b]$, is equal to zero.

Often we also encounter (for example, in problems of quantum mechanics) the Tschebyscheff-Hermite differential equation with

$$k(x) = e^{-x^2}, \qquad \rho = e^{x^2}, \qquad q = 0, \qquad a = -\infty, \qquad b = \infty;$$

the Tschebyscheff-Laguerre differential equation with

$$k(x) = xe^{-x}, \qquad \rho(x) = e^{-x}, \qquad q = 0, \qquad a = 0, \qquad b = \infty;$$

and others. The boundary-value problems of these differential equations define very important classes of special functions (cylindrical functions, spherical functions, Tschebyscheff-Hermite polynomials, Tschebyscheff-Laguerre polynomials, etc.).

2. Formulation of the boundary-value problems in the case $k(a) = 0$

We shall now investigate some general properties of the equation

$$L[y] = \frac{d}{dx}\left[k(x)\frac{dy}{dx}\right] - q(x)y = 0 \tag{A-1.6}$$

for the case $k(a) = 0$. We therefore assume that $k(x)$ in the neighborhood of the point a possesses the form $k(x) = (x - a)\varphi(x)$, where $\varphi(x)$ is a continuous function and $\varphi(a) \neq 0$.

If $q(x)$ is replaced by $q(x) - \lambda\rho(x)$ in the differential equation, then we recognize that all of the following considerations are also valid for the equation

$$L[y] + \lambda\rho y = 0. \tag{A-1.1}$$

First the following lemma shall be proved concerning the solution of Eq. (A-1.6).

Lemma 1. If in Eq. (A 1.6) the coefficient $k(x)$ for $x = a$ is equal to 0 $[k(x) = (x - a)\varphi(x), \varphi(a) \neq 0]$ and if a solution $y_1(x)$ of this equation for $x = a$ has a finite limit value $[y_1(a) \neq \infty]$, then every other solution $y_2(x)$ of Eq. (A-1.6) which is linearly independent of $y_1(x)$ is infinite for $x = a$.

Proof. If a solution $y_1(x)$ of the linear differential equation (A-1.6) is known, then any other solution $y_2(x)$ linearly independent of $y_1(x)$ can be found by quadrature. Namely, if we consider Eq. (A-1.6) for $y_1(x)$ and $y_2(x)$, multiply these with y_2 and y_1, respectively, and subtract the equation so obtained, we then obtain

$$y_2\frac{d}{dx}\left(k\frac{dy_1}{dx}\right) - y_1\frac{d}{dx}\left(k\frac{dy_2}{dx}\right) - \frac{d}{dx}[k(y_2y_1' - y_1y_2')] = 0.$$

Hence, it follows that the Wronski determinant of the functions $y_1(x)$ and $y_2(x)$ is equal to $C[k(x)]$:

$$y_1\frac{dy_2}{dx} - y_2\frac{dy_1}{dx} = \frac{C}{k(x)}.$$

The constant C is different from 0 provided $y_1(x)$ and $y_2(x)$ are linearly independent. After division of the last equation by y_1^2, we obtain

$$\frac{d}{dx}\left(\frac{y_2(x)}{y_1(x)}\right) - \frac{C}{ky_1^2}$$

or

$$y_2(x) - y_1(x)\left[\int_{x_0}^{x} \frac{C\,d\alpha}{k(\alpha)y_1^2(\alpha)} + C_1\right]. \qquad (A\ 1.7)$$

The integration constant C_1 depends on the selection of the point x_0.

The function $y_1(x)$, which has a finite value for $x \quad a$, can be equal to zero there. We assume (in particular, see Lemma 3) that

$$y_1(x) = (x-a)^n z_1(x)\,, \qquad z_1(a) \neq 0\,, \quad n \geqq 0\,.$$

The point x_0 in Eq. (A-1.7) is selected so that

$$z_1(x) \neq 0\,, \cdot \quad a < x \leqq x_0$$

[or $y_1(x) \neq 0$, $a \leqq x \leqq x_0$, provided $n = 0$]. Then for $y_2(x)$ in the interval $a < x \leqq x_0$, the representation

$$y_2(x) = (x-a)^n z_1(x)\left\{C_1 - \int_x^{x_0} \frac{1}{(x-a)^{2n+1}}\left[\frac{C}{\varphi(x)z_1^2(x)}\right]dx\right\}$$

holds. Hence, by application of the mean value theorem to the integral on the right, we obtain for $y_2(x)$ the expression

$$y_2(x) = (x-a)^n z_1(x)\left\{C_1 - A\int_x^{x_0} \frac{dx}{(x-a)^{2n+1}}\right\},$$

$$A = \frac{C}{\varphi(\bar{x})z_1^2(\bar{x})}\,, \qquad (x \leqq \bar{x} \leqq x_0)\,,$$

or by calculation of the integral,

$$y_2(x) = \begin{cases} (x-a)^n z_1(x)\left[C_1 + A\dfrac{(x_0-a)^{2n}}{2n} - A\dfrac{(x-a)^{-2n}}{2n}\right] & \text{for } n > 0\,, \\ y_1(x)[C_1 - A\ln(x_0-a) + A\ln(x-a)] & \text{for } n = 0\,. \end{cases} \qquad (A\text{-}1.8)$$

In both cases, by passage to the limit $x \to a$ we obtain $\lim_{x\to a} y_2(x) = +\infty$, whereby Lemma 1 is proved; for $n > 0$, there follows from Eq. (A-1.8)

$$\lim_{x\to a}(x-a)^n y_2(x) = -\frac{C}{2n\varphi(a)z_1(a)}\,. \qquad (A\text{-}1.9)$$

Formulas (A-1.8) and (A-1.9) permit the following sharpening of Lemma 1:

If under the assumptions of Lemma 1 the bounded solution $y_1(x)$ of Eq. (A-1.6) for $x = a$ is different from 0 [$y_1(a) \neq 0$], then $y_2(x)$ at the point $x = a$ has a logarithmic singularity

$$y_2(x) \sim \ln(x-a)\,.$$

If $y_1(x)$ at the point $x = a$ has a zero of multiplicity n,

$$[y_1(x) = (x-a)^n z_1(x)\,, \qquad z_1(a) \neq 0]\,,$$

then $y_2(x)$ for $x = a$ has a pole of order n, $y_2(x) \sim (x-a)^{-n}$.

Lemma 2. If in Eq. (A-1.6) the coefficient $k(x)$ for $x = a$ is equal to zero, $[k(x) = (x - a)\varphi(x),\ \varphi(a) \neq 0]$, while the coefficient $q(x)$ as $x \to a$ either remains bounded or approaches $+\infty$, then for each solution $y_1(x)$ of Eq. (A-1.6) bounded at the point $x = a$, we find

$$\lim_{x \to a} k(x) y_1'(x) = 0 .$$

Proof. We consider first the case that $q(x)$ as $x \to a$ remains bounded. By integration of Eq. (A-1.6) over $[x, x_1]$ with $x < x_1$, we obtain

$$k(x)\frac{dy_1}{dx} = k(x)\frac{dy_1}{dx}\bigg|_{x=x_1} - \int_x^{x_1} q(\alpha)\, y(\alpha)\, d\alpha = Q(x) , \qquad (x < x_1) . \quad \text{(A 1.10)}$$

Hence, we find that $Q(x)$ for $a \leqq x \leqq x_1$ is a continuous function. If we now carry out the passage to the limit as $x \to a$, we see that the limit value

$$C = \lim_{x \to a} k(x)\, y_1'(x) , \qquad (C = Q(a))$$

exists. Now we shall prove that $C = Q(a) = 0$. For this purpose, we express $y_1(x)$ through $Q(x)$:

$$\begin{aligned} y_1(x) &= y_1(x_2) - \int_x^{x_2} \frac{Q(\alpha)}{k(\alpha)}\, d\alpha \\ &= y_1(x_2) - \int_x^{x_2} \frac{Q(\alpha)}{(\alpha - a)\,\varphi(\alpha)}\, d\alpha , \qquad (a \leqq x \leqq x_2) . \end{aligned} \quad \text{(A-1.11)}$$

From this expression, we can immediately see that in the case $Q(a) \neq 0$ as $x \to a$, the function $y_1(x)$ approaches infinity.

However, this contradicts the boundedness of $y_1(x)$ for $x = a$. Therefore, Lemma 2 for the case of a bounded $q(x)$ is proved. Still remaining to be considered is the case that $q(x)$ for $x \to a$ tends toward $+\infty$. Expression (A-1.10) also holds in this case. Let the function $q(x)$ in the interval $a \leqq x \leqq x_1$ be positive. Then the function $y_1(x)$ in this interval is monotone.[46]

Moreover, in a fixed interval $a \leqq x \leqq x_2$, it keeps the same sign and as $x \to a$ has a limit value $y_1(a)$. Consequently, $Q(x)$ in this interval must be a monotone function which, for $x = a$, has a finite or infinite limit value. From expression (A-1.11) it now follows again that $y_1(x) \to \infty$, when $Q(a) \neq 0$. Thus Lemma 2 is completely proved.

For the next discussion we note that from Eq. (A 1.10) and from the condition $Q(x) \to 0$ as $x \to a$, the expression

$$k(x_1)\, y_1'(x_1) = \int_a^{x_1} q(\alpha)\, y_1(\alpha)\, d\alpha \quad \text{(A-1.12)}$$

follows, irrespective of whether $q(x)$ is bounded or not, $(q(x) > 0)$.

[46] Thus if we set $\xi = \int_x^{x_1} dx/k(x)$, then Eq. (A-1.6) can be written in the form

$$y'' - kqy = 0 , \qquad (k > 0 ,\ q > 0) .$$

However, it follows that y cannot have positive maxima (negative minima) since at such points $y'' \leqq 0$ $(y'' \geqq 0)$, but necessarily $y > 0$ $(y < 0)$.

Lemma 3. If $y_1(x)$ is a bounded solution of Eq. (A-1.6) for $x = a$ and $q(x)$ is continuous in the interval $a \leqq x \leqq x_1$, then $y_1(a) \neq 0$ and $y_1'(a) = [q(a)y_1(a)]'\varphi(a)$.

If $q(x)$ possesses the form $q(x) = [q_1(x)]/(x - a)$, where $q_1(x)$ is continuous and $q_1(a) > 0$, then $y_1(a) = 0$.

Proof. The second part of the lemma follows, on the basis of expression (A-1.12), directly from the convergence of the integrals

$$\int_a^x q(\alpha)\, y_1(\alpha)\, d\alpha = \int_a^x \frac{q_1(\alpha)}{(\alpha - a)}\, y_1(\alpha)\, d\alpha \ .$$

However, this is only possible when $y_1(a) = 0$.

If $q(x)$ is a continuous function in $a \leqq x \leqq x_1$, then, since x_1 is arbitrarily selected in expression (A-1.12), we obtain

$$y_1'(x) = \frac{1}{\varphi(x)} \frac{1}{x - a} \int_a^x q(\alpha)\, y(\alpha)\, d\alpha \ . \tag{A-1.13}$$

Consequently, the function $y_1'(x)$ must be bounded in a fixed interval $a \leqq x \leqq x_2$ and there the limit value of $y_1(x)$ as $x \to a$ must exist. Now if we carry out the passage to the limit as $x \to a$ in Eq. (A-1.13), then we obtain

$$\lim_{x \to a} y_1'(x) = \frac{q(a)\, y_1(a)}{\varphi(a)} = y_1'(a) \ .$$

Consequently, $y_1(a) = 0$ implies $y_1'(a) = 0$. We shall show, then, that $y_1(x) \equiv 0$ [the direct application of the uniqueness theorem is not possible here, since $x = a$ is a singular point of Eq. (A-1.6)]. Let $|q(x)| \leqq \bar{q}$ and $|\varphi(x)| \geqq \varphi > 0$ for $a \leqq x \leqq x_2$.

We consider the interval $(a, a + h)$, where h is a sufficiently small number, so that $h(\bar{q}/\varphi) < 1$ and $a + h < x_2$. Thus from Eq. (A-1.13) it follows that

$$y_1(x) = \int_a^x \left(\frac{1}{\varphi(\alpha_1)} \frac{1}{(\alpha_1 - a)} \int_a^{\alpha_1} q(\alpha_2)\, y(\alpha_2)\, d\alpha_2 \right) d\alpha_1 \ , \qquad (y_1(a) = 0) \ .$$

Let the function $y_1(x)$ in the interval $(a, a + h)$ assume its maximum $y_1(x_0) = A$ at $x = x_0$. By application of the mean value theorem we obtain

$$y_1(x_0) = A \leqq A\bar{q}\frac{1}{\varphi}(x_0 - a) < A \ ,$$

whereby our assertion is proved.

Lemma 4. If the function $q(x)$ has the form $q(x) = [q_1(x)]/(x - a)$, where $q_1(x) > 0$ for $a \leqq x \leqq x_1$, then every solution of Eq. (A-1.6) bounded for $x = a$ has the form $y_1(x) = (x - a)^\nu z(x)$, where $\nu = \sqrt{[q_1(a)]/[\varphi(a)]}$ and $z(x)$ is a continuous and differentiable function in $a \leqq x \leqq x_1$ with $z(a) \neq 0$.

Proof. If we set $y_1(x) = (z - a)^\nu z(x)$, where ν is a fixed number, then from expression (A-1.12) we obtain

$$\nu k(x)(x - a)^{\nu - 1} z(x) + k(x)(x - a)^\nu \frac{dz}{dx} = \int_a^x q(\alpha)\, y(\alpha)\, d\alpha \ . \tag{A-1.14}$$

By partial integration, therefore,

$$\int_{a+\epsilon}^{x} q(\alpha)\,y(\alpha)\,d\alpha = \int_{a+\epsilon}^{x}\left[\frac{q_1(\alpha)}{(\alpha-a)}(\alpha-a)^{\nu}\right]z(\alpha)\,d\alpha$$
$$= z(\alpha)u(\alpha)\Big|_{a+\epsilon}^{x} - \int_{a+\epsilon}^{x} u'(\alpha)z'(\alpha)\,d\alpha\ ,$$

where

$$u(\alpha) = \int_a^{\alpha}\frac{q_1(\alpha_1)}{(\alpha_1-a)}(\alpha_1-a)^{\nu}\,d\alpha_1$$
$$= \frac{q_1(a)}{\nu}(\alpha-a)^{\nu} + \int_a^{\alpha}\frac{q_1(\alpha)-q_1(a)}{(\alpha_1-a)}(\alpha_1-a)^{\nu}\,d\alpha_1 = \frac{q_1(a)}{\nu}(\alpha-a)^{\nu} + u_1(\alpha)\ .$$

Next, we conclude from the expression $z'(\alpha)u'(\alpha) = y_1(\alpha)[q_1(a)/\nu] + [u_1(\alpha)/(\alpha-a)^{\nu}]$ that $\lim_{\alpha\to a} z(\alpha)u(\alpha) = 0$. Since $u_1(\alpha)/(\alpha-a)^{\nu}$ for $\alpha\to a$ is bounded, by Lemma 3, $\lim_{\alpha\to a} y_1(\alpha) = 0$.

Further, for $z'(x)$ we obtain from Eq. (A-1.14)

$$z'(x) = z(x)\left[-\frac{\nu}{(x-a)} + \frac{u(x)}{k(x-a)^{\nu}}\right] - \int_a^x u(\alpha)z'(\alpha)\,d\alpha$$
$$= z(x)U(x) - \int_a^x u(\alpha)z'(\alpha)\,d\alpha\ . \qquad \text{(A-1.15)}$$

Now if we choose $\nu = \sqrt{q_1(a)/\varphi(a)}$, then it follows that the function

$$U(x) = -\frac{\nu}{(x-a)} + \frac{u(x)}{k(x-a)^{\nu}} = \frac{u_1(x)}{k(x-a)^{\nu}}$$

is continuous and differentiable for $a \leqq x \leqq x_1$. By differentiation of Eq. (A-1.15),

$$z''(x) = [U(x) - u(x)]z'(x) + U'(x)z(x) \qquad \text{(A-1.16)}$$

follows.

The point $x = a$ is not a singular point of Eq. (A-1.15). Thus it follows that the function $z(x)$ for $a \leqq x \leqq x_1$ is continuous and possesses there a continuous derivative. By passage to the limit in Eq. (A-1.15) as $x \to a$, we find further that

$$z'(a) = z(a)U(a)\ . \qquad \text{(A-1.17)}$$

Formula (A-1.17) shows that $z(a) \neq 0$, since otherwise $z(a) = z'(a) = 0$, which would mean that $z \equiv 0$.

We point out that the function $U(x)$ for $x = a$ is continuous if we assume the (once) differentiability of the function $q_1(x)$, whereas in order to guarantee the differentiability of $U'(x)$, the differentiability of $\varphi(x)$ and the twice differentiability of $q_1(x)$ is assumed. Lemma 4, however, can be proved without the differential equation (A-1.16). Instead we can consider Eq. (A-1.15) as an integral equation for the function $z(x)$. To do this it is necessary to require only the continuity of the function $U(x)$.

This lemma permits the following conclusion in the formulation of boundary-value problems for the equations

$$\boldsymbol{L}[y] + \lambda\rho y = 0 \qquad \text{and} \qquad \boldsymbol{L}[y] = 0\ ,$$

when the coefficient function $k(x)$ is equal to zero at one or both end points of the interval $[a, b]$ under consideration. If $k(a) = 0$, then for $x = a$ we impose the requirement that the eigenfunctions be bounded. This does not require that they assume a prescribed value for $x = a$.

In this manner we arrive at the following boundary value problem:

Determine the eigenvalues and eigenfunctions of the equation

$$L[y] = \frac{d}{dx}\left[k(x)\frac{dy}{dx}\right] - q(x)y = -\lambda\rho y\,, \qquad (a < x < b)\,,$$

in which $k(x) > 0$ for $x > a$ and $k(a) = 0$ which satisfies the boundary condition

$$y(b) = 0$$

and the boundedness condition for $x = a$.

If $k(a) = 0$ and $k(b) = 0$, then at both end points of the interval $[a, b]$ the boundedness is required.

Now we shall formulate the general properties of the eigenfunctions and eigenvalues of this boundary-value problem:

1. There exist infinitely many eigenvalues $\lambda_1 \leqq \lambda_2 \leqq \cdots$, which correspond to the eigenfunctions

$$y_1(x), y_2(x), \ldots, y_n(x), \ldots .$$

2. For $q \geqq 0$, all eigenvalues are nonnegative:

$$\lambda_n \geqq 0\,.$$

3. Two eigenfunctions $y_n(x)$ and $y_m(x)$, which correspond to the different eigenvalues λ_n and λ_m, are orthogonal to each other with respect to density $\rho(x)$:

$$\int_a^b y_n(x)\, y_m(x)\rho(x)\, dx = 0\,.$$

4. The expansion theorem. A function $f(x)$, which has a continuous first and a piecewise, continuous second derivative, can be expanded in an absolutely and uniformly convergent series with respect to the eigenfunctions $y_n(x)$:

$$f(x) = \sum_{n=1}^{\infty} f_n y_n(x)\,, \quad \text{where} \quad f_n = \frac{\int_a^b f(x)y_n(x)\rho(x)\,dx}{\int_a^b y_n^2(x)\rho(x)\,dx}$$

when it satisfies the boundary conditions of the problem. If $k(a) = 0$, we obtain

$$|f(a)| < \infty \quad \text{for } q(a) < \infty\,,$$

$$f(a) = 0 \quad \text{for } q(x) = \frac{q_1(x)}{x - a}\,, \quad (q_1(a) \neq 0)\,.$$

Properties 2 and 3 were proved in Section 2-3 by means of Green's formula. Here, the boundedness of $y_n(x)$ for $x = a$ was used as well as the equation $\lim_{x \to a} k(x)\, y_n'(x) = 0$ proved above. The proof for properties 1 and 4 is usually carried out by using the theory of integral equations. We shall not go into this proof now. In order for properties 1 and 4 to hold, it is

sufficient to require that the function $k(x)$ including its derivative be continuous, while $q(x)$ either is continuous or possesses the form $q_1(x)/(x-a)$, where $q_1(x)$ is continuous. These conditions are satisfied for the classes of special functions to be investigated in the following discussion.

Our boundary-value problem is equivalent to the integral equation

$$y(x) = \lambda \int_a^b G(x, \xi)\, y(\xi)\rho(\xi)\, d\xi \,.$$

By means of the substitutions

$$\varphi(x) = \sqrt{\rho(x)}\, y(x) \,, \qquad K(x, \xi) = \sqrt{\rho(x)}\, G(x, \xi)\sqrt{\rho(\xi)} \,;$$

this substitution leads to the integral equation

$$\varphi(x) = \lambda \int_a^b K(x, \xi)\varphi(\xi)\, d\xi$$

with the symmetrical kernel $K(x, \xi)$. Here $G(x, \xi)$ is the Green's function for the equation $L[y] = 0$. For the case $k(a) = 0$, $k(b) \neq 0$, $y(b) = 0$, it is defined by the following conditions:

1. For fixed ξ, $G(x, \xi)$ is a continuous function of x.
2. The first derivative dG/dx has the discontinuity at $x = \xi$

$$\left.\frac{dG}{dx}\right|_{x=\xi-0}^{x=\xi+0} = -\frac{1}{k(\xi)}$$

or

$$G'(\xi + 0, \xi) - G'(\xi - 0, \xi) = -\frac{1}{k(\xi)} \,.$$

3. With the exception of $x = \xi$, $L[G] = 0$ everywhere.
4. $G(x, \xi)$ satisfies the boundary conditions

$$G(a, \xi) < \infty \,, \qquad G(b, \xi) = 0 \,.$$

A-2. CYLINDRICAL FUNCTIONS

1. The cylindrical functions

In the solution of many problems of mathematical physics one often encounters the ordinary differential equation

$$\frac{d^2y}{dx^2} + \frac{1}{x}\frac{dy}{dx} + \left(1 - \frac{n^2}{x^2}\right)y = 0$$

or

$$\frac{1}{x}\frac{d}{dx}\left(x\frac{dy}{dx}\right) + \left(1 - \frac{n^2}{x^2}\right)y = 0 \,, \tag{A-2.1}$$

the so-called differential equation of cylindrical functions of the nth order. Usually, we also speak of the Bessel differential equation of the nth order.

Characteristic problems (see Chapters V, VI, and VII) which lead to cylindrical functions are represented by the boundary-value problems for the equation

$$\Delta u + k^2 u = 0 \,, \tag{A-2.2}$$

if the region considered is exterior or interior to a circle (the region outside or inside of a cylinder for three independent variables). In polar coordinates, Eq. (A-2.2) assumes the form

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 u}{\partial \varphi^2}+k^2u=0\,. \tag{A-2.3}$$

If we set $u = R\Phi$ and if the separation of variables is carried out in Eq. (A-2.3), we obtain

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{dR}{dr}\right)+\left(k^2-\frac{\lambda}{r^2}\right)R=0$$

and

$$\Phi''+\lambda\Phi=0\,.$$

From the periodicity condition for $\Phi(\varphi)$, it follows that $\lambda = n^2$, where n is an integer. Now if we set $x = kr$, then we arrive at the differential equation of cylindrical functions

$$\frac{1}{x}\frac{d}{dx}\left(x\frac{dy}{dx}\right)+\left(1-\frac{n^2}{x^2}\right)y=0\,,\qquad R(r)=y(kr)$$

or

$$y''+\frac{1}{x}y'+\left(1-\frac{n^2}{x^2}\right)y=0\,.$$

For the case of the solution of the wave equation (A-2.2) with radial (cylindrical) symmetry, the Bessel differential equation of *zero*th order is obtained:

$$\frac{1}{x}\frac{d}{dx}\left(x\frac{dy}{dx}\right)+y=0\qquad\text{or}\qquad y''+\frac{1}{x}y'+y=0\,.$$

1. *Power Series.* The Bessel differential equation of the νth order

$$y''+\frac{1}{x}y'+\left(1-\frac{\nu^2}{x^2}\right)y=0 \tag{A-2.1'}$$

or

$$x^2y''+xy'+(x^2-\nu^2)y=0 \tag{A-2.1''}$$

(let ν be an arbitrary real or complex number whose real part we can assume to be nonnegative) possesses a singularity for $x = 0$. Consequently, we shall seek the solution in the form of a power series[47]

$$y(x)=x^{\sigma}(a_0+a_1x+a_2x^2+\cdots+a_kx^k+\cdots) \tag{A-2.4}$$

which begins with x^{σ}. The characteristic exponent σ remains to be determined. We substitute the series (A-2.4) into Eq. (A-2.1'). All the coefficients of the powers $x^{\sigma}, x^{\sigma+1}, \ldots, x^{\sigma+k}$ thus occurring must be equal to zero. For the deter-

[47] See W. W. Stepanov, *Textbook of Differential Equations*, VEB Deutscher Verlag der Wissenschaften, Berlin, 1956.

mination of σ and the coefficients a_k, we obtain the system of equations

$$\begin{aligned} a_0(\sigma^2 - \nu^2) &= 0 , \\ a_1[(\sigma + 1)^2 - \nu^2] &= 0 , \\ a_2[(\sigma + 2)^2 - \nu^2] + a_0 &= 0 , \\ &\cdots \\ a_k[(\sigma + k)^2 - \nu^2] + a_{k-2} &= 0 , \\ (k &= 2, 3, \ldots) . \end{aligned} \tag{A-2.5}$$

Since we can assume that $a_0 \neq 0$, then from the first equation of system (A 2.5) we find the characteristic exponents

$$\sigma^2 - \nu^2 = 0 \qquad \text{or} \qquad \sigma = \pm\nu . \tag{A-2.6}$$

The other k equations represent a system of equations for a_k. We write the kth equation ($k > 1$) of system (A-2.5) in the form

$$(\sigma + k + \nu)(\sigma + k - \nu)a_k + a_{k-2} = 0 . \tag{A-2.7}$$

In the following consideration we now exclude the case that $\sigma + \nu$ or $\sigma - \nu$ (and, correspondingly, -2ν) is equal to a negative integer.

Then, from the second equation of system (A-2.5) there follows

$$a_1 = 0 \tag{A-2.8}$$

because of the exponents (A 2.6).

Furthermore, Eq. (A-2.7) for all following coefficients a_k ($k > 1$) yields the recursion formula

$$a_k = -\frac{a_{k-2}}{(\sigma + k + \nu)(\sigma + k - \nu)} , \tag{A-2.9}$$

from which we can determine a_k when a_{k-2} is known.

From formulas (A 2.9) and (A-2.8) it is apparent that all odd coefficients are equal to zero. If ν is real, then for $\rho = -\nu$ the solution at the point $x = 0$ is infinite.

Now let us consider the case $\sigma = \nu$. From formula (A-2.9) it follows, therefore, that each even coefficient can be expressed in terms of the previous even coefficients:

$$a_{2m} = -a_{2m-2}\frac{1}{2^2 m(m + \nu)} . \tag{A-2.10}$$

By repeated application of this formula a_{2m} can be expressed through a_0:

$$a_{2m} = (-1)^m \frac{a_0}{2^{2m} m!(\nu + 1)(\nu + 2)\cdots(\nu + m)} . \tag{A-2.11}$$

Now we use the property of the Γ function,[48] that is,

$$\Gamma(s + 1) = s\Gamma(s) \quad \cdots = s(s - 1)\cdots(s - n)\Gamma(s - n) .$$

If s is an integer, then

$$\Gamma(s + 1) = s! .$$

The coefficient a_0 is still arbitrary. If $\nu \neq n$ where $n > 0$ denotes an integer, and if we set

$$a_0 = \frac{1}{2^\nu \Gamma(\nu + 1)} \tag{A 2.12}$$

and consider the above designated property of the Γ function, then we obtain

$$a_{2k} = (-1)^k \frac{1}{2^{2k+\nu}\Gamma(k+1)\Gamma(k+\nu+1)} . \tag{A 2.13}$$

However, if $\sigma = -\nu$ with $\nu \neq n$, and if we set

$$a_0 = \frac{1}{2^{-\nu}\Gamma(-\nu+1)} , \tag{A 2.12'}$$

where $n > 0$ is an integer, then we obtain the relation

$$a_{2k} = (-1)^k \frac{1}{2^{2k-\nu}\Gamma(k+1)\Gamma(k-\nu+1)} . \tag{A-2.14}$$

Series (A 2.4), corresponding to the case $\sigma = \nu \geqq 0$ with coefficients (A 2.12) and (A-2.13),

[48] The function $\Gamma(s)$ defined by the formula

$$\Gamma(s) = \int_0^\infty e^{-x} x^{s-1} dx \quad \text{for } s > 0$$

is called the Gamma function. The variable s must also assume complex values:

$$s = s_0 + is_1 , \qquad s_0 > 0 .$$

We list the basic properties of the Γ function:

(a) Partial integration in the formula for $\Gamma(s)$ gives

$$\Gamma(s+1) = \int_0^\infty e^{-x} x^s dx = -e^{-x} x^s \Big|_0^\infty + s\int_0^\infty e^{-x} x^{s-1} dx = s\Gamma(s) ,$$

and thus

$$\Gamma(s+1) = s\Gamma(s) .$$

(b) For $s = 1$, there follows directly

$$\Gamma(1) = 1 .$$

(c) If $s = n$ is an integer, then from (a) and (b) we find

$$\Gamma(n+1) = n! .$$

(d) For $s = \frac{1}{2}$,

$$\Gamma(s) - \Gamma(\tfrac{1}{2}) = \int_0^\infty \frac{e^{-x}}{\sqrt{x}} dx - 2\int_0^\infty e^{-\xi^2} d\xi = \sqrt{\pi} ,$$

and thus

$$\Gamma(\tfrac{1}{2}) = \sqrt{\pi} .$$

(e) The functional relation

$$\Gamma(s+1) = s\Gamma(s)$$

also permits us to define the Γ function for negative values of s. It is defined by

$$\Gamma(0) = \frac{\Gamma(s+1)}{s}\Big|_{s=0} = \infty$$

and

$$\Gamma(-n) = \frac{\Gamma(-n+1)}{-n} = \cdots = (-1)^n \frac{\Gamma(0)}{n!} = \infty .$$

$$J_\nu(x) \quad \sum_{k=0}^{\infty} (-1)^k \frac{1}{\Gamma(k+1)\Gamma(k+\nu+1)} \left(\frac{x}{2}\right)^{2k+\nu}, \tag{A 2.15}$$

is called the Bessel function of the first kind and the νth order. The series

$$J_{-\nu}(x) \quad \sum_{k=0}^{\infty} \frac{(-1)^k}{\Gamma(k+1)\Gamma(k-\nu+1)} \left(\frac{x}{2}\right)^{2k-\nu}, \tag{A 2.16}$$

which corresponds to the case $\sigma = -\nu$, represents a second solution of (A 2.1). It is linearly independent of $J_\nu(x)$. Series (A 2.15) and (A-2.16) obviously converge in the entire x-plane.

We shall now consider the case where ν is to equal to $\frac{1}{2}$ of an integer.

Therefore, let $\nu^2 = (n+\frac{1}{2})^2$ where $n = 0$ is an integer. Then, if in (A 2.5) we set $\sigma = n+\frac{1}{2}$, we obtain

$$2(n+1)a_1 = 0,$$

$$k(k+2n+1)a_k + a_{k-2} = 0, \qquad (k > 1).$$

Consequently,

$$a_1 = 0, \qquad a_k = -\frac{a_{k-2}}{k(k+2n+1)}.$$

By repeated application of this formula we find

$$a_{2k} = \frac{(-1)^k a_0}{2\cdot 4\cdots(2k)(2n+3)(2n+5)\cdots(2n+2k+1)}.$$

For $\nu = n+\frac{1}{2}$ we therefore obtain formula (A 2.11).

Furthermore, if we set

$$a_0 = \frac{1}{2^{n+(1/2)}\Gamma(n+\frac{3}{2})},$$

then formula (A-2.13) results.

Now let

$$\sigma = -n-\frac{1}{2}.$$

Then Eqs. (A 2.5) for the coefficients a_k assume the form

$$a_1\cdot 1(-2n) = 0,$$

$$k(k-1-2n)a_k + a_{k-2} = 0.$$

As stated earlier, all odd coefficients $a_1, a_3, \ldots, a_{2n-1}$ are equal to zero, while for a_{2n+1} we obtain the equation

$$0\cdot a_{2n+1} + a_{2n-1} = 0,$$

which is satisfied for each value of a_{2n+1}. For $k > n$, a_{2k+1} can be calculated from the equation

$$a_{2k+1} = \frac{(-1)^{k-n} a_{2n+1}}{(2n+3)(2n+5)\cdots 2\cdot 4\cdots(2k-2n)}.$$

For $a_{2n+1} = 0$,

$$a_0 = \frac{1}{2^{-n}\,{}^{1/2}\Gamma(\frac{1}{2} - n)}$$

gives formula (A-2.14).

Therefore, the case $\nu = \pm(n + \frac{1}{2})$ requires no change in the definition of the functions $J_\nu(x)$. Formulas (A-2.15) and (A-2.16) retain their validity.

Formula (A-2.16) defines $J_\nu(x)$ only for fractional values of ν, since the definition of a_0 according to formula (A-2.12) loses its meaning for negative integer values of $\nu = -n$. We shall now extend formula (A-2.16) to integer values $\nu = n$. Since $\Gamma(k - n + 1) = \infty$ for $k \leqq k_0 = n - 1$, and if for convenience we set $1/\Gamma(k - n + 1) = 0$, the summation in formula (A-2.16) actually begins for $k = k_0 + 1 = n$. If in formula (A-2.16) the summation index is changed by setting $k = n + k'$, then we obtain

$$J_{-n}(x) = (-1)^n \sum_{k'=0}^{\infty} \frac{(-1)^{k'}}{\Gamma(k' + n + 1)\Gamma(k' + 1)} \left(\frac{x}{2}\right)^{2k'-n} = (-1)^n J_n(x) ,$$

since the summation now begins with $k' = 0$.

In the following discussion, we give, as an example, the series for the Bessel functions of the first kind of *zero*th order ($n = 0$) and of first order ($n = 1$):

$$J_0(x) = 1 - \left(\frac{x}{2}\right)^2 + \frac{1}{(2!)^2}\left(\frac{x}{2}\right)^4 - \frac{1}{(3!)^2}\left(\frac{x}{2}\right)^6 + \cdots ,$$

$$J_1(x) = \frac{x}{2} - \frac{1}{2!}\left(\frac{x}{2}\right)^3 + \frac{1}{2!\,3!}\left(\frac{x}{2}\right)^5 - \cdots .$$

Exact tables exist for the functions $J_0(x)$ and $J_1(x)$ which often occur in applications.[49] Towards the end of this chapter we give the graphical representations for $J_0(x)$ and $J_1(x)$.

The functions $J_n(x)$ and $J_{-n}(x)$, as already stated, are linearly dependent for integer values n:

$$J_{-n}(x) = (-1)^n J_n(x) .$$

For noninteger values of ν, $J_\nu(x)$, and $J_{-\nu}(x)$ are linearly independent. At the point $x = 0$, $J_\nu(x)$ possesses a zero, while $J_{-\nu}(x)$ has a pole of order ν there. Consequently, if ν is not an integer, every solution $J_\nu(x)$ of Bessel's equation (A-2.1) can be represented as a linear combination of the functions $J_\nu(x)$ and $J_{-\nu}(x)$:

$$y_\nu(x) = C_1 J_\nu(x) + C_2 J_{-\nu}(x) .$$

If a bounded solution of Eq. (A-2.1) is sought, then $C_2 = 0$:

$$y_\nu(x) = C_1 J_\nu(x) .$$

[49] Almost all tables for the special functions also give tables for the Bessel functions of the first kind. For example, E. Jahnke and F. Emde in *Tables of Bessel Functions* (Dover, New York, 1945) give five-place values of $J_0(x)$ and $J_1(x)$ for x from 0 to 14.9.

2. *Recursion formulas.* Between the Bessel functions of the first kind the following relations

$$\frac{d}{dx}\left(\frac{J_\nu(x)}{x^\nu}\right) = -\frac{J_{\nu+1}(x)}{x^\nu}, \tag{A-2.17}$$

$$\frac{d}{dx}(x^\nu J_\nu(x)) = x^\nu J_{\nu-1}(x) \tag{A-2.18}$$

are valid. These formulas can be derived directly by differentiation of the corresponding series for the Bessel functions. We shall prove the validity of expression (A-2.17) as an example. First we have

$$x^\nu \frac{d}{dx}\left(\frac{J_\nu(x)}{x^\nu}\right) = x^\nu \frac{1}{2^\nu} \sum_{k=1}^{\infty} (-1)^k \frac{\frac{1}{2}\left(\frac{x}{2}\right)^{2k-1} 2k}{k!\,\Gamma(k+\nu+1)}$$

$$= \sum_{k=1}^{\infty} (-1)^k \frac{1}{\Gamma(k)\Gamma(k+\nu+1)} \left(\frac{x}{2}\right)^{2k+\nu-1}$$

In the last sum, the summation is extended over k from 1 to ∞. In place of k we now introduce the summation index $l = k - 1$, which extends from 0 to ∞. Then

$$x^\nu \frac{d}{dx}\left(\frac{J_\nu(x)}{x^\nu}\right) = -\sum_{l=0}^{\infty} (-1)^l \frac{1}{\Gamma(l+1)\Gamma[l+(\nu+1)+1]} \left(\frac{x}{2}\right)^{[2l+(\nu+1)]} = -J_{\nu+1}(x),$$

whereby (A-2.17) is proved. The validity of formula (A-2.18) is proved analogously.

We shall now establish two important special cases of the recursion formulas. For $\nu = 0$, we find from formula (A-2.17) that

$$J_0'(x) = -J_1(x), \tag{A-2.19}$$

and for $\nu = 1$, from formula (A-2.18), that

$$[xJ_1(x)]' = xJ_0(x) \qquad \text{or} \qquad xJ_1(x) = \int_0^x \xi J_0(\xi)\, d\xi. \tag{A-2.20}$$

Finally, we establish a few more recursion formulas which relate $J_\nu(x)$, $J_{\nu+1}(x)$, and $J_{\nu-1}(x)$ with each other. By differentiation of formulas (A-2.17) and (A-2.18), we first obtain

$$\frac{\nu J_\nu(x)}{x} - J_\nu'(x) = J_{\nu+1}(x), \tag{A-2.17'}$$

$$\frac{\nu J_\nu(x)}{x} + J_\nu'(x) = J_{\nu-1}(x). \tag{A-2.18'}$$

Addition and subtraction of formulas (A-2.17') and (A-2.18') then gives

$$\begin{aligned} J_{\nu+1}(x) + J_{\nu-1}(x) &= \frac{2\nu}{x} J_\nu(x), \\ J_{\nu+1}(x) - J_{\nu-1}(x) &= -2J_\nu'(x). \end{aligned} \tag{A-2.21}$$

By means of formulas (A-2.21), $J_{\nu+1}(x)$ can be calculated if $J_\nu(x)$ and $J_{\nu-1}(x)$ are known:

$$J_{\nu+1}(x) = -J_{\nu-1}(x) + \frac{2\nu J_\nu(x)}{x} . \tag{A-2.21'}$$

3. *The Bessel Function of the first kind of order* $n + \frac{1}{2}$. First we shall derive simple representations of the functions $J_{1/2}(x)$ and $J_{-1/2}(x)$:

$$J_{1/2}(x) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\,\Gamma(\frac{3}{2} + m)} \left(\frac{x}{2}\right)^{1/2+2m} , \tag{A-2.22}$$

$$J_{-1/2}(x) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\,\Gamma(\frac{1}{2} + m)} \left(\frac{x}{2}\right)^{-(1/2)+2m} \tag{A-2.23}$$

From the properties of the Γ function we find

$$\begin{aligned} \Gamma(\tfrac{3}{2} + m) &= \frac{1\cdot 3\cdot 5 \cdots (2m+1)}{2^{m+1}} \Gamma(\tfrac{1}{2}) , \\ \Gamma(\tfrac{1}{2} + m) &= \frac{1\cdot 3 \cdots (2m-1)}{2^m} \Gamma(\tfrac{1}{2}) , \end{aligned} \tag{A-2.24}$$

where

$$\Gamma(\tfrac{1}{2}) = \sqrt{\pi} .$$

By substituting Eqs. (A-2.24) into Eqs. (A-2.22) and (A-2.23), we obtain

$$J_{1/2}(x) = \sqrt{\frac{2}{\pi x}} \sum_{m=0}^{\infty} \frac{(-1)^m}{(2m+1)!} x^{2m+1} , \tag{A-2.25}$$

$$J_{-1/2}(x) = \sqrt{\frac{2}{\pi x}} \sum_{m=0}^{\infty} \frac{(-1)^m}{(2m)!} x^{2m} . \tag{A-2.26}$$

The sum in Eq. (A-2.25) obviously represents the expansion of $\sin x$, and the sum in Eq. (A-2.26) that of $\cos x$ in powers of x. In this manner the functions $J_{1/2}(x)$ and $J_{-1/2}(x)$ can be expressed by elementary functions:

$$J_{1/2}(x) = \sqrt{\frac{2}{\pi x}} \sin x , \tag{A-2.27}$$

$$J_{-1/2}(x) = \sqrt{\frac{2}{\pi x}} \cos x . \tag{A-2.28}$$

Now we consider the function $J_{n+1/2}(x)$ for integer n. From formula (A-2.21') it follows that

$$\begin{aligned} J_{3/2}(x) &= \frac{1}{x} J_{1/2}(x) - J_{-1/2}(x) = \sqrt{\frac{2}{\pi x}} \left(-\cos x + \frac{\sin x}{x}\right) \\ &= \sqrt{\frac{2}{\pi x}} \left[\sin\left(x - \frac{\pi}{2}\right) + \frac{1}{x} \cos\left(x - \frac{\pi}{2}\right)\right] , \\ J_{5/2}(x) &= \sqrt{\frac{2}{\pi x}} \left\{-\sin x + \frac{3}{x}\left[\sin\left(x - \frac{\pi}{2}\right) + \frac{1}{x} \cos\left(x - \frac{\pi}{2}\right)\right]\right\} \\ &= \sqrt{\frac{2}{\pi x}} \left\{\sin(x - \pi)\left(1 - \frac{3}{x^2}\right) + \cos(x - \pi) \cdot \frac{3}{x}\right\} . \end{aligned}$$

By repeated application of formula (A-2.21') we finally obtain

$$J_{n+1/2}(x) = \sqrt{\frac{2}{\pi x}}\left\{\sin\left(x - \frac{n\pi}{2}\right)P_n\left(\frac{1}{x}\right) + \cos\left(x - \frac{n\pi}{2}\right)Q_n\left(\frac{1}{x}\right)\right\}, \quad \text{(A-2.29)}$$

where $P_n(1/x)$ is a polynomial of the nth degree and $Q_n(1/x)$ is a polynomial of the $(n-1)$th degree in $1/x$. Thus $P_n(0) = 1$ and $Q_n(0) = 0$.

4. *Asymptotic representation of cylindrical functions for large x.* We shall show that each real cylindrical function for large x can be represented in the form

$$y_\nu(x) = \gamma_\infty \frac{\sin(x + \delta_\infty)}{\sqrt{x}} + O\left(\frac{1}{x^{3/2}}\right). \quad \text{(A-2.30)}$$

Here, $\gamma_\infty \neq 0$, δ_∞ is a fixed constant and $0(1/x^{3/2})$ denotes those terms[50] whose order is not smaller than $1/x^{3/2}$.

Now we set

$$y = \frac{v(x)}{\sqrt{x}} \quad \text{(A-2.31)}$$

and calculate the derivatives

$$y' = -0.5x^{-3/2}v + x^{-1/2}v',$$
$$y'' = x^{-1/2}v'' - x^{-3/2}v' + 0.75x^{-5/2}v.$$

If we put these expressions into Bessel's equation, we obtain the equation

$$v'' + \left(1 - \frac{\nu^2 - \frac{1}{4}}{x^2}\right)v = 0. \quad \text{(A-2.32)}$$

This equation appears as a special case of the equation

$$v'' + v + \rho(x)v = 0, \quad \text{(A-2.33)}$$

where

$$\rho(x) = O\left(\frac{1}{x^2}\right). \quad \text{(A-2.34)}$$

Now we set

$$v = \gamma \sin(x + \delta), \qquad v' = \gamma \cos(x + \delta), \quad \text{(A-2.35)}$$

where $\gamma(x)$ and $\delta(x)$ are determinate functions of x. For all x, $\gamma(x)$ must not be equal to zero, otherwise v and v' vanish simultaneously, which would mean that $v(x)$ is identically equal to zero. From expressions (A-2.35) and (A-2.33), we obtain

$$v' = \gamma \cos(x + \delta) = \gamma' \sin(x + \delta) + \gamma(\delta' + 1)\cos(x + \delta),$$
$$v'' = \gamma' \cos(x + \delta) - \gamma(\delta' + 1)\sin(x + \delta) = -(1 + \rho)\gamma \sin(x + \delta).$$

From these expressions we find

$$\delta' = \rho \sin^2(x + \delta), \qquad \delta' = O\left(\frac{1}{x^2}\right). \quad \text{(A-2.36)}$$

$$\frac{\gamma'}{\gamma} = -\frac{\delta'}{\operatorname{tg}(x + \delta)} = \rho \sin(x + \delta)\cos(x + \delta) = O\left(\frac{1}{x^2}\right). \quad \text{(A-2.37)}$$

[50] O represents the so-called Landau symbol.

We shall now show that γ and δ, as $x \to \infty$, approach a definite limit value. First we write

$$\delta(x) = \delta(a) - \int_x^a \delta'(s)\, ds ,$$

from which, because of Eq. (A-2.36), it follows that the limit value $\lim_{a\to\infty} \delta(a) = \delta_\infty$ exists and

$$\delta(x) = \delta_\infty + O\left(\frac{1}{x}\right) . \tag{A-2.38}$$

Analogously, from Eq. (A-2.37) it follows that

$$\gamma(x) = \gamma_\infty\left(1 + O\left(\frac{1}{x}\right)\right) , \tag{A-2.39}$$

where $\gamma_\infty \neq 0$.

Moreover, each solution of Eq. (A-2.36) and, consequently, each solution of Eq. (A-2.32), as $x \to \infty$, has the form

$$v(x) = \gamma_\infty \sin(x + \delta_\infty) + O\left(\frac{1}{x}\right) , \tag{A-2.40}$$

whereby the validity of the asymptotic representation (A-2.30) for an arbitrary, real cylinderical function $y_\nu(x)$ is proved.

We shall now show that there cannot exist two distinct real cylindrical functions whose asymptotic behavior coincides. To do this, we assume the existence of two different cylindrical functions $\bar{y}_\nu(x)$ and $\bar{\bar{y}}_\nu(x)$ for which

$$\bar{\gamma}_\infty = \bar{\bar{\gamma}}_\infty , \qquad \bar{\delta}_\infty = \bar{\bar{\delta}}_\infty . \tag{A-2.41}$$

The difference

$$\tilde{y}_\nu(x) = \bar{y}_\nu(x) - \bar{\bar{y}}_\nu(x) \not\equiv 0$$

of these functions is likewise a real cylindrical function which, because of (A-2.41), shows the following asymptotic behavior:

$$\tilde{y}_\nu(x) = O\left(\frac{1}{x^{3/2}}\right) .$$

However, this contradicts formula (A-2.30), which is valid for each cylindrical function. Consequently, $\tilde{y}_\nu(x) \equiv 0$ and $\bar{y}_\nu(x) \equiv \bar{\bar{y}}_\nu(x)$.

The values of the constants γ_∞ and δ_∞ can be calculated by additional considerations which give

$$\gamma_\infty = \sqrt{\frac{2}{\pi}} \quad \text{for all } \nu.$$

In Section A-2.1(3) we derived formula (A-2.29) for $\nu = n + \frac{1}{2}$ from which we obtained

$$J_{n+1/2}(x) = \sqrt{\frac{2}{\pi x}} \sin\left(x - \frac{n\pi}{2}\right) + O\left(\frac{1}{x^{3/2}}\right) . \tag{A-2.42}$$

Previously the following asymptotic formula for $J_\nu(x)$ was derived, if $\nu = n$ is an integer:

$$J_\nu(x) = \sqrt{\frac{2}{\pi x}} \cos\left(x - \frac{\pi}{2}\nu - \frac{\pi}{4}\right) + O\left(\frac{1}{x^{3/2}}\right). \tag{A-2.43}$$

By investigations based on the representation of the functions $J_\nu(x)$ and $J_{-\nu}(x)$ by means of contour integrals in the complex plane, we can show that formula (A-2.43) is also valid for arbitrary ν. Here we obtain

$$J_{-\nu}(x) = \sqrt{\frac{2}{\pi x}} \cos\left(x + \frac{\pi}{2}\nu - \frac{\pi}{4}\right) + O\left(\frac{1}{x^{3/2}}\right). \tag{A-2.44}$$

2. Boundary-value problems for the Bessel differential equation

The simplest boundary-value problem for the Bessel differential equation (we consider this in the interval $[0, r_0]$) is connected with the problem of the eigenvibrations of a circular membrane:

$$\Delta_2 v + \lambda v = 0, \qquad \Delta_2 v = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 v}{\partial \varphi^2}, \tag{A-2.45}$$

$$v(r, \varphi)|_{r=r_0} = 0. \tag{A-2.46}$$

If we set $v(r, \varphi) = R(r)\Phi(\varphi)$ and carry out the separation of variables, we obtain

$$\Phi'' + \nu\Phi = 0, \tag{A-2.47}$$

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{dR}{dr}\right) + \left(\lambda - \frac{\nu}{r^2}\right)R = 0, \qquad R(r_0) = 0. \tag{A-2.48}$$

If we let $\Phi(\varphi)$ be periodic, then $\nu = n^2$, where n is an integer. The function $R(r)$, therefore, satisfies the Bessel equation

$$L[R] + \lambda r R = 0, \qquad \left(L[R] = \frac{d}{dr}\left(r\frac{dR}{dr}\right) - \frac{n^2}{r}R\right) \tag{A-2.49}$$

with the boundary condition

$$R(r_0) = 0 \tag{A-2.50}$$

and the boundedness condition

$$|R(0)| < \infty \tag{A-2.51}$$

at the point $r = 0$. Then if we set

$$x = \sqrt{\lambda}\, r, \qquad y(x) = R(r) = R\left(\frac{x}{\sqrt{\lambda}}\right), \tag{A-2.52}$$

we obtain the equation

$$\frac{1}{x}\frac{d}{dx}\left(x\frac{dy}{dx}\right) + \left(1 - \frac{n^2}{x^2}\right)y = 0 \tag{A-2.53}$$

with the auxiliary conditions

$$y(\sqrt{\lambda}\, r_0) = 0, \tag{A-2.54}$$

$$|y(0)| < \infty. \tag{A-2.55}$$

Thus we find

$$y(x) = AJ_n(x) . \tag{A-2.56}$$

Because of the boundary condition $y(r_0\sqrt{\lambda}) = 0$, $r_0\sqrt{\lambda} = \mu$,

$$J_n(\mu) = 0 \tag{A-2.57}$$

is valid. This transcendental equation obviously possesses infinitely many real roots $\mu_1^{(n)}, \mu_2^{(n)}, \ldots, \mu_m^{(n)}, \ldots$[51] that is, Eq. (A-2.45) has infinitely many eigenvalues

$$\lambda_m^{(n)} = \left(\frac{\mu_m^{(n)}}{r_0}\right)^2 , \qquad (m = 1, 2, \ldots) , \tag{A-2.58}$$

which correspond to the eigenfunctions

$$R(r) = AJ_n\left(\frac{\mu_m^{(n)}}{r_0} r\right) \tag{A-2.59}$$

of the boundary-value problem (A-2.45), (A-2.46). From the type and manner in which the eigenfunctions have been constructed, it can be seen that all nontrivial solutions of the boundary-value problem considered are represented by formula (A-2.59).

From the above consideration of the general theory of equation

$$L[y] + \lambda \rho y = 0$$

follows the orthogonality of the system

$$\left\{J_n\left(\frac{\mu_m^{(n)}}{r_0} r\right)\right\}$$

of eigenfunctions with respect to density r:

$$\int_0^{r_0} J_n\left(\frac{\mu_{m_1}^{(n)}}{r_0} r\right) J_n\left(\frac{\mu_{m_2}^{(n)}}{r_0} r\right) r\, dr = 0 \quad \text{for } m_1 \neq m_2 . \tag{A-2.60}$$

We shall calculate the norm of the eigenfunction $R_1(r) = J_n(\alpha_1 r)$ with $\alpha_1 = \mu_m^{(n)}/r_0$. For this we consider the function $R_2(r) = J_n(\alpha_2 r)$, where α_2 is an arbitrary parameter.

The functions $R_1(r)$ and $R_2(r)$ satisfy the equations

$$\frac{d}{dr}\left(r\frac{dR_1}{dr}\right) + \left(\alpha_1^2 r - \frac{n^2}{r}\right) R_1 = 0 ,$$

$$\frac{d}{dr}\left(r\frac{dR_2}{dr}\right) + \left(\alpha_2^2 r - \frac{n^2}{r}\right) R_2 = 0 .$$

Thus $R_1(r_0) = 0$, whereas $R_2(r)$ still does not satisfy this boundary condition. Now if we subtract the second equation from the first, after which we multiply it with $R_2(r)$ and $R_1(r)$ and then integrate with respect to r over the interval $[0, r_0]$, we obtain

$$(\alpha_1^2 - \alpha_2^2)\int_0^{r_0} rR_1(r)R_2(r)\, dr + [r(R_2R_1' - R_1R_2')]\Big|_0^{r_0} = 0$$

[51] On p. 000 a table of the roots of the equation $J_0(\mu) = 0$ is given. The first root is $\mu_1(0) = 2.4048$.

and, therefore,

$$\int_0^{r_0} R_1 R_2 r\,dr = \frac{r_0 J_n(\alpha_2 r_0)\alpha_1 J_n'(\alpha_1 r_0) - r_0 J_n(\alpha_1 r_0)\alpha_2 J_n'(\alpha_2 r_0)}{\alpha_1^2 - \alpha_2^2}$$

$$= -\frac{r_0 J_n(\alpha_2 r_0)\alpha_1 J_n'(\alpha_1 r_0)}{\alpha_1^2 - \alpha_2^2} .$$

Now we carry out the passage to the limit $\alpha_2 \to \alpha_1$. Then for the norm of the expression we obtain

$$N = \int_0^{r_0} r R_1^2(r)\,dr = \frac{r_0^2}{2}[J_n'(\alpha_1 r_0)]^2$$

or

$$\int_0^{r_0} J_n^2\left(\frac{\mu_m^{(n)}}{r_0} r\right) r\,dr = \frac{r_0^2}{2}[J_n'(\mu_m^{(n)})]^2 . \tag{A 2.61}$$

In particular, the norm of the function $J_0((\mu_m^{(0)}/r_0)r)$ equals

$$\int_0^{r_0} J_0^2\left(\frac{\mu_m^{(0)}}{r_0} r\right) r\,dr = \frac{r_0^2}{2} J_1^2(\mu_m^{(0)}) . \tag{A-2.62}$$

Because of the general properties of the eigenfunctions of the boundary-value problem and on the basis of the above observations the following expansion theorem holds:

Theorem. Every twice differentiable function $f(r)$ which for $r = 0$ is bounded and for $r = r_0$ is equal to zero can be expanded in an absolutely and uniformly convergent series

$$f(r) = \sum_{m=1}^{\infty} A_m J_n\left(\frac{\mu_m^{(n)}}{r_0} r\right) ,$$

in which

$$A_m = \frac{\int_0^{r_0} f(r) J_n\left(\frac{\mu_m^{(n)}}{r_0} r\right) r\,dr}{N_m} .$$

$$N_m = \frac{r_0^2}{2}[J_n'(\mu_m^{(n)})]^2 .$$

The second boundary-value problem for the Bessel equation

$$L(R) + \lambda r R = 0 ,$$
$$R'(r_0) = 0 ,$$
$$R(0) < \infty$$

is solved analogously. The eigenfunctions and eigenvalues are again expressed by formulas (A-2.59) and (A-2.58). Here, by $\mu_m^{(n)}$ is understood the mth root of the equation

$$J_n'(\mu) = 0 .$$

The eigenfunctions of the problems are orthogonal to each other with respect to density r [see formula (A-2.60)]. For its norm we have

$$\int_0^{r_0} J_n^2\left(\frac{\mu_m^{(n)} r}{r_0}\right) r\,dr = \frac{r_0^2}{2}\left[1 - \frac{n^2}{(\mu_m^{(n)})^2}\right] J_n^2(\mu_m^{(n)}) .$$

The third boundary-value problem is also solved analogously. In this case for the determination of $\mu_m^{(n)}$ there results an equation of the form

$$J_n'(\mu) = hJ_n(\mu) .$$

3. The different types of cylindrical functions

1. *Hankel functions.* In addition to the Bessel functions of the first kind $J_\nu(x)$, there are still other types of special solutions of the Bessel equations with great significance in applications. The Hankel functions of the first and second kind $H_\nu^{(1)}(x)$ and $H_\nu^{(2)}(x)$ belong to them. With these functions, which are conjugate to each other, one can treat complex solutions of the Bessel equation. From the standpoint of physical applications, the asymptotic behavior for large values of the argument

$$H_\nu^{(1)}(x) = \sqrt{\frac{2}{\pi x}} e^{i(x - \pi/2\cdot\nu - \pi/4)} + \cdots , \tag{A 2.63}$$

$$H_\nu^{(2)}(x) = \sqrt{\frac{2}{\pi x}} e^{-i(x - i\pi/2\cdot\nu - \pi/4)} + \cdots \tag{A 2.64}$$

appear as basic characteristics of the Hankel functions. The dots here denote those terms which are of higher order smaller than $1/x$. If the real and imaginary parts in the Hankel functions are separated, then we obtain

$$H_\nu^{(1)}(x) = J_\nu(x) + iN_\nu(x) , \tag{A-2.65}$$

$$H_\nu^{(2)}(x) = J_\nu(x) - iN_\nu(x) , \tag{A-2.66}$$

where the functions

$$J_\nu(x) = \tfrac{1}{2}[H_\nu^{(1)}(x) + H_\nu^{(2)}(x)] , \tag{A-2.65'}$$

$$N_\nu(x) = \frac{1}{2i}[H_\nu^{(1)}(x) - H_\nu^{(2)}(x)] \tag{A-2.66'}$$

have asymptotic behavior

$$J_\nu(x) = \sqrt{\frac{2}{\pi x}} \cos\left(x - \frac{\pi}{2}\nu - \frac{\pi}{4}\right) + \cdots , \tag{A-2.67}$$

$$N_\nu(x) = \sqrt{\frac{2}{\pi x}} \sin\left(x - \frac{\pi}{2}\nu - \frac{\pi}{4}\right) + \cdots , \tag{A-2.68}$$

which follows from formulas (A-2.63) and (A-2.64).

As will be shown later [see Section A-2.3(3)], with the functions $J_\nu(x)$ introduced here, we can actually treat the Bessel functions of the first kind which were considered in Section A-2.1. The imaginary part $N_\nu(x)$ of the νth Hankel function is called the Neumann function or the cylindrical function of the second kind of νth order.

Formulas (A-2.65) and (A-2.66) represent the relation between the Hankel, Bessel, and Neumann functions. They correspond to the relation that the complex exponential functions have with the sine and cosine functions (Euler's formula). The asymptotic formulas (A-2.63), (A-2.64), (A-2.67), and (A-2.68) underlie this analogy.

From the investigation of the solutions of the wave equation

$$u_{tt} = a^2(u_{xx} + u_{yy}) ,$$

we see that the amplitude $v(x, y)$ of a forced vibration

$$u(x, y, t) = v(x, y)e^{i\omega t}$$

satisfies the vibration equation

$$v_{xx} + v_{yy} + k^2 v = \Delta v + k^2 v = 0 , \qquad \left(k^2 = \frac{\omega^2}{a^2}\right) .$$

If the solution of the vibration equation is radially symmetric, that is, $v(x, y) = v(r)$, then the function $v(kr)$, as was noted in Section A-2.1, satisfies Bessel's equation of *zero*th order.

Therefore, the functions

$$H_0^{(1)}(kr)e^{i\omega t} = \sqrt{\frac{2}{\pi kr}}\, e^{i(\omega t + kr)} \frac{1}{\sqrt{i}} + \cdots \qquad \text{(A-2.69)}$$

and

$$H_0^{(2)}(kr)e^{i\omega t} = \sqrt{\frac{2}{\pi kr}}\, e^{i(\omega t - kr)} \sqrt{i} + \cdots \qquad \text{(A-2.70)}$$

$$\left.\begin{array}{l}\\ \\ \end{array}\right\} , \qquad (\sqrt{i} = e^{i(\pi/4)}) ,$$

represent solutions of the vibration equation which exhibits the character of a cylindrical wave. The function $H_0^{(2)}(kr)e^{i\omega t}$ corresponds to a divergent spherical wave, while the function $H_0^{(1)}(kr)e^{i\omega t}$ corresponds to a convergent spherical wave.[52]

2. *Hankel and Neumann functions.* As was noted in Section A-2.1, each solution of the Bessel equation can be expressed through the functions $J_\nu(x)$ and $J_{-\nu}(x)$ with noninteger order ν. We shall now determine the relationship between the functions $H_\nu^{(1)}$, $H_\nu^{(2)}$, N_ν and J_ν, $J_{-\nu}$.

Since each solution of the Bessel equation for noninteger ν can be represented as a linear combination of the functions $J_\nu(x)$ and $J_{-\nu}(x)$,

$$H_\nu^{(1)}(x) = C_1 J_\nu(x) + C_2 J_{-\nu}(x) , \qquad \text{(A-2.71)}$$

where C_1 and C_2 are constants still to be determined. For the principal term of the asymptotic representations there obviously exists the analogous equation

$$\begin{aligned}\sqrt{\frac{2}{\pi x}}\, e^{i(x - (\pi/2)\nu - \pi/4)} &= C_1 \sqrt{\frac{2}{\pi x}} \cos\left(x - \frac{\pi}{2}\nu - \frac{\pi}{4}\right) \\ &\quad + C_2 \sqrt{\frac{2}{\pi x}} \cos\left(x + \frac{\pi}{2}\nu - \frac{\pi}{4}\right) .\end{aligned} \qquad \text{(A-2.72)}$$

Next we transform the argument of the second summand to the form $(x - (\pi/2)\nu - \pi/4)$:

$$\begin{aligned}\cos\left(x + \frac{\pi}{2}\nu - \frac{\pi}{4}\right) &= \cos\left[\left(x - \frac{\pi}{2}\nu - \frac{\pi}{4}\right) + \pi\nu\right] \\ &= \cos\left(x - \frac{\pi}{2}\nu - \frac{\pi}{4}\right)\cos\pi\nu - \sin\left(x - \frac{\pi}{2}\nu - \frac{\pi}{4}\right)\sin\pi\nu .\end{aligned}$$

[52] We take the time factor $e^{-i\omega t}$, so that $H_0^{(1)}(kr)e^{-i\omega t}$ corresponds to the divergent and $H_0^{(2)}(kr)e^{-i\omega t}$ to the convergent spherical wave.

Now if we divide both sides of Eq. (A-2.72) by $\sqrt{2/(\pi x)}$, then by using Euler's formula we obtain

$$\cos\left(x-\frac{\pi}{2}\nu-\frac{\pi}{4}\right)+i\sin\left(x-\frac{\pi}{2}\nu-\frac{\pi}{4}\right)$$
$$=(C_1+C_2\cos\pi\nu)\cos\left(x-\frac{\pi}{2}\nu-\frac{\pi}{4}\right)-C_2\sin\pi\nu\sin\left(x-\frac{\pi}{2}\nu-\frac{\pi}{4}\right),$$

from which it follows that

$$C_1+C_2\cos\pi\nu=1\,,\qquad -C_2\sin\pi\nu=i$$

or

$$C_2=\frac{1}{i\sin\pi\nu}\,,$$
$$C_1=-\frac{\cos\pi\nu-i\sin\pi\nu}{i\sin\pi\nu}=-C_2e^{-i\pi\nu}\,. \tag{A-2.73}$$

By substituting (A-2.73) in (A-2.71), we find

$$H_\nu^{(1)}(x)=-\frac{1}{i\sin\pi\nu}[J_\nu(x)e^{-i\pi\nu}-J_{-\nu}(x)]\,. \tag{A-2.74}$$

Analogously, we obtain

$$H_\nu^{(2)}(x)=\frac{1}{i\sin\pi\nu}[J_\nu(x)e^{i\pi\nu}-J_{-\nu}(x)]\,. \tag{A-2.75}$$

Now if we use formula (A-2.66′), by which the function $N_\nu(x)$ is defined, then from formulas (A-2.74) and (A-2.75) we obtain

$$N_\nu(x)=\frac{J_\nu(x)\cos\pi\nu-J_{-\nu}(x)}{\sin\pi\nu}\,. \tag{A-2.76}$$

Formulas (A-2.74), (A-2.75), and (A-2.76) were derived for noninteger ν. If one carries out the passage to the limit $\nu\to n$ in this formula, then for the Hankel and Neumann functions for integer $\nu=n$, we obtain

$$H_n^{(1)}(x)=J_n(x)+i\frac{1}{\pi}\left[\left(\frac{\partial J_\nu}{\partial\nu}\right)_{\nu=n}-(-1)^n\left(\frac{\partial J_{-\nu}}{\partial\nu}\right)_{\nu=n}\right], \tag{A 2.74′}$$

$$H_n^{(2)}(x)=J_n(x)-i\frac{1}{\pi}\left[\left(\frac{\partial J_\nu}{\partial\nu}\right)_{\nu=n}-(-1)^n\left(\frac{\partial J_{-\nu}}{\partial\nu}\right)_{\nu=n}\right], \tag{A-2.75′}$$

$$N_n(x)=\frac{1}{\pi}\left[\left(\frac{\partial J_\nu}{\partial\nu}\right)_{\nu=n}-(-1)^n\left(\frac{\partial J_{-\nu}}{\partial\nu}\right)_{\nu=n}\right]. \tag{A-2.76′}$$

By using the power series expansion of J_ν and $J_{-\nu}$, analogous series representations are derived both for $N_\nu(x)$ and also for $H_\nu^{(1)}(x)$ and $H_\nu^{(2)}(x)$.

Formulas (A-2.74) and (A-2.75) can be considered as the analytic definition of the Hankel functions. There are still other methods for the definition of Hankel functions. Thus in Section A-2.6 the Hankel functions will be represented by contour integrals.

For $\nu = n - \frac{1}{2}$, the Hankel and Neumann functions can be expressed by elementary functions. For example, for $\nu = \frac{1}{2}$, we obtain

$$N_{1/2}(x) = -J_{-1/2}(x) = -\sqrt{\frac{2}{\pi x}}\cos x = \sqrt{\frac{2}{\pi x}}\sin\left(x - \frac{\pi}{2}\right),$$

$$H_{1/2}^{1}(x) = J_{1/2}(x) - iN_{1/2}(x) = \sqrt{\frac{2}{\pi x}}\left[\cos\left(x - \frac{\pi}{2}\right) + i\sin\left(x - \frac{\pi}{2}\right)\right]$$

$$= \sqrt{\frac{2}{\pi x}}e^{i(x-\pi/2)},$$

$$H_{1/2}^{2}(x) = J_{1/2}(x) - iN_{1/2}(x) = \sqrt{\frac{2}{\pi x}}\left[\cos\left(x - \frac{\pi}{2}\right) - i\sin\left(x - \frac{\pi}{2}\right)\right]$$

$$= \sqrt{\frac{2}{\pi x}}e^{-i(x-\pi/2)}.$$

3. *Bessel functions with imaginary arguments.* Cylindrical functions can be considered not only for real, but also for complex values of the argument. In this section we shall investigate Bessel functions of the first kind with purely imaginary arguments.

For this we replace the argument x by ix in the series expansion for $J_\nu(x)$ and obtain

$$J_\nu(ix) = i^\nu \sum_{k=0}^{\infty} \frac{(-1)^{2k}}{\Gamma(k-1)\Gamma(k+\nu+1)}\left(\frac{x}{2}\right)^{2k-\nu} = i^\nu I_\nu(x), \qquad \text{(A-2.77)}$$

where

$$I_\nu(x) = \sum_{k=0}^{\infty} \frac{1}{\Gamma(k+1)\Gamma(k+\nu+1)}\left(\frac{x}{2}\right)^{2k-\nu} \qquad \text{(A-2.78)}$$

is a real function connected with $J_\nu(ix)$ by the relation

$$I_\nu(x) = i^{-\nu}J_\nu(ix) \quad \text{or} \quad I_\nu(x) = e^{-\pi\nu i/2}J_\nu(ix)$$

In particular, for $\nu = 0$,

$$I_0(x) = J_0(ix) = 1 - \left(\frac{x}{2}\right)^2 + \frac{1}{(2!)^2}\left(\frac{x}{2}\right)^4 + \frac{1}{(3!)^2}\left(\frac{x}{2}\right)^6 + \cdots. \qquad \text{(A-2.79)}$$

From the series (A-2.78) it is apparent that the functions $I_\nu(x)$ are monotone increasing functions. From the asymptotic representation (A-2.67), for large values of x for $I_\nu(x)$ the following asymptotic representation holds:

$$I_\nu(x) \approx \sqrt{\frac{1}{2\pi x}}e^x. \qquad \text{(A-2.80)}$$

Bessel functions with imaginary argument are solutions of the equation

$$y'' + \frac{1}{x}y' - \left(1 + \frac{\nu^2}{x^2}\right)y = 0. \qquad \text{(A-2.81)}$$

In particular, $I_0(x)$ satisfies the equation

$$y'' + \frac{1}{x}y' - y = 0. \qquad \text{(A-2.82)}$$

In addition to the functions $I_\nu(x)$, we also consider functions defined for purely imaginary argument with the help of Hankel functions

$$K_\nu(x) = \frac{1}{2}\pi i e^{\pi\nu i/2} H_\nu^{(1)}(ix) . \tag{A-2.83}$$

Furthermore, as will be shown below, $K_\nu(x)$ also assumes real values for real x-values. This follows from the formula

$$K_\nu(x) = \frac{1}{2}\int_{-\infty}^{\infty} e^{-x\cosh\eta - \nu\eta}\, d\eta , \tag{A-2.84}$$

which we shall derive in Section A-2.6. From the asymptotic representation for $H_\nu^{(1)}$ we find

$$K_\nu(x) = \sqrt{\frac{\pi}{2x}}\, e^{-x} + \cdots . \tag{A-2.85}$$

Formulas (A-2.85) and (A-2.80) show that as $x \to \infty$, $K_\nu(x)$ decreases exponentially, while $J_\nu(x)$ increases exponentially as $x \to \infty$. From this follows the linear independence of these functions and therefore the possibility of representing an arbitrary solution of equation (A-2.81) as a linear combination

$$y = AI_\nu(x) + BK_\nu(x) .$$

In particular, if y is bounded at infinity, then necessarily $A = 0$ and $y = BK_\nu(x)$.

The function

$$K_0(x) = \int_0^{\infty} e^{-x\cosh\eta}\, d\eta$$

also has great significance.

4. *The function* $K_0(x)$. We now show that for the function $K_0(x)$ the integral representation

$$K_0(x) = \int_0^{\infty} e^{-x\cosh\xi}\, d\xi , \qquad (x > 0) \tag{A-2.86}$$

holds.

First, we easily see that the integral

$$F(x) = \int_0^{\infty} e^{-x\cosh\xi}\, d\xi \tag{A-2.86'}$$

satisfies the equation

$$L(y) = y'' + \frac{1}{x}y' - y = 0 . \tag{A-2.87}$$

Namely,

$$L(F) = \int_0^{\infty} e^{-x\cosh\xi}\left(\cosh^2\xi - \frac{1}{x}\cosh\xi - 1\right) d\xi$$

$$= \int_0^{\infty} e^{-x\cosh\xi}\sinh^2\xi\, d\xi - \frac{1}{x}\int_0^{\infty} e^{-x\cosh\xi}\cosh\xi\, d\xi = S_1 - S_2 .$$

By partial integration of the two summands, we obtain

$$S_2 = \frac{1}{x}\int_0^\infty e^{-x\cosh\xi}\cosh\xi\,d\xi = \frac{\sinh\xi}{x}e^{-x\cosh\xi}\Big|_0^\infty + \int_0^\infty e^{-x\cosh\xi}\sinh^2\xi\,d\xi = S_1 ,$$

from which

$$L(F) = 0$$

follows. If we set $\cosh\xi = \eta$, then the integral (A-2.86′) for $F(x)$ is transformed into the form

$$F(x) = \int_1^\infty \frac{e^{-x\eta}}{\sqrt{\eta^2 - 1}}\,d\eta .$$

From this formula we can infer the behavior of the function $F(x)$ as $x \to \infty$. Hence, by one more change of variable

$$x(\eta - 1) = \xi ,$$

we obtain

$$F(x) = \frac{e^{-x}}{\sqrt{x}}\int_0^\infty \frac{e^{-\xi}}{\sqrt{\xi((\xi/x) + 2)}}\,d\xi = \frac{e^{-x}}{\sqrt{x}}F_1(x) .$$

As $x \to \infty$, then

$$\lim F_1(x) = \int_0^\infty \frac{e^{-\xi}}{\sqrt{2\xi}}\,d\xi = \frac{2}{\sqrt{2}}\int_0^\infty e^{-t^2}\,dt = \frac{\sqrt{\pi}}{\sqrt{2}} , \qquad (t = \sqrt{\xi}) .$$

Consequently, for large values of x, we obtain

$$F_1(x) = \frac{\sqrt{\pi}}{\sqrt{2}}(1 + \varepsilon) ,$$

where $\varepsilon \to 0$ as $x \to \infty$. Hence, the asymptotic representation

$$F(x) = \sqrt{\frac{\pi}{2x}}e^{-x} + \cdots \qquad \text{(A-2.88)}$$

follows. The dots here denote those terms which are small and of higher order. The function $F(x)$ introduced by means of the integral (A-2.86′) represents a solution of Eq. (A-2.87) bounded at infinity. Therefore, we have

$$F(x) = BK_0(x) .$$

By comparison of the asymptotic representations for $K_0(x)$ and $F(x)$, it is shown that $B = 1$. Consequently,

$$K_0(x) = \int_0^\infty e^{-x\cosh\xi}\,d\xi , \qquad (x > 0) . \qquad \text{(A-2.86)}$$

We shall now investigate the behavior of $K_0(x)$ as $x \to 0$.

Here we write the integral

$$K_0(x) = F(x) = \int_1^\infty \frac{e^{-x\eta}}{\sqrt{\eta^2 - 1}}\,d\eta$$

in the form

$$K_0(x) = \int_x^\infty \frac{e^{-\lambda}}{\sqrt{\lambda^2 - x^2}}\, d\lambda\,, \qquad (x\eta = \lambda)\,,$$

and split it into three summands:

$$K_0(x) = \int_x^A \frac{d\lambda}{\sqrt{\lambda^2 - x^2}} + \int_x^A \frac{(e^{-\lambda} - 1)\, d\lambda}{\sqrt{\lambda^2 - x^2}} + \int_A^\infty \frac{e^{-\lambda}\, d\lambda}{\sqrt{\lambda^2 - x^2}}$$

(A is a fixed auxiliary constant). Obviously, the first summand is equal to

$$\ln \frac{A + \sqrt{A^2 - x^2}}{x} = -\ln x + \cdots\,,$$

while the second and third summands, as $x \to 0$, remain bounded. From this follows

$$K_0(x) = -\ln x + \cdots = \ln \frac{1}{x} + \cdots\,, \tag{A-2.89}$$

where the dots denote the terms which remain bounded at $x = 0$. Consequently, $K_0(x)$ represents a solution of Eq. (A-2.87) which exhibits a logarithmic singularity at the point $x = 0$ and which vanishes exponentially as $x \to \infty$.

The following problem explains the physical significance of the function $K_0(x)$. At the origin of coordinates lies a stationary source of an unstable gas of intensity Q_0. The stationary diffusion process is accompanied by a decomposition of the gas and is described by the equation

$$\Delta u - \kappa^2 u = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \varphi^2} - \kappa^2 u = 0\,, \qquad \left(\kappa^2 = \frac{\beta}{D^2}\right). \tag{A-2.90}$$

Here β is the decomposition coefficient and D is the diffusion coefficient. The Green's function of this equation is circular by symmetry and satisfies the equation

$$\frac{1}{x}\frac{d}{dx}\left(x\frac{du}{dx}\right) - u = 0\,, \qquad (x = \kappa r)\,.$$

Moreover, Green's function has a logarithmic singularity at the origin of coordinates and is bounded at infinity. Consequently, Green's function is proportional to $K_0(\kappa r)$:

$$\bar{G} = AK_0(\kappa r)\,. \tag{A-2.91}$$

For the determination of the factor A, we use the source equation

$$\lim_{\varepsilon \to 0} \int_{K_\varepsilon} \left(-D\frac{\partial u}{\partial r}\right) ds = Q_0\,. \tag{A-2.92}$$

The integral standing on the left side of this equation expresses the diffusion flow through the surface K_ε of the sphere with radius ε and source point as center. If we replace u by the function $\bar{G} = AK_0(\kappa r)$ in this equation and consider the logarithmic singularity of $K_0(x)$ at $x = 0$, we obtain

$$\lim_{\varepsilon \to 0}\left\{-\int_{K_\varepsilon} D\frac{\partial \bar{G}}{\partial r}\, dS\right\} = \lim_{\varepsilon \to 0}\left\{D\, 2\pi\varepsilon A\frac{1}{\varepsilon}\right\} = 2\pi AD = Q_0\,.$$

From this we find

$$A = \frac{Q_0}{2\pi D}$$

and

$$\bar{G} = \frac{Q_0}{2\pi D} K_0(\kappa r) . \tag{A-2.93}$$

The integral formula (A-2.86) for $K_0(x)$ can be derived from simple physical considerations.

We shall now treat the nonstationary problem of diffusion of a gas with decomposition. At the origin of coordinates we place a source with constant intensity Q_0 which acts continuously from time $t = 0$. We shall assume that at the initial moment $t = 0$, the concentration of the gas is everywhere equal to zero. The concentration $u(x, y, t)$ must satisfy the equation

$$D\,\Delta u - \beta u = u_t \tag{A-2.94}$$

and the corresponding auxiliary conditions. With the help of the substitution

$$u = \tilde{u} e^{-\beta t}$$

Eq. (A-2.94) takes the form of the ordinary diffusion equation

$$D\,\Delta \tilde{u} = \tilde{u}_t .$$

Green's function for this equation has the form

$$\tilde{G} = \frac{1}{(2\sqrt{\pi D(t-\tau)})^2} e^{-[r^2/(4D(t-\tau))]} , \qquad (D = a^2) .$$

Consequently, for Green's function of Eq. (A-2.94), we have

$$G = \frac{Q}{(2\sqrt{\pi D(t-\tau)})^2} e^{-[r^2/(4D(t-\tau))]-\beta(t-\tau)} .$$

Green's function for a source of intensity Q_0 acting from $t = 0$ to the time t is then given by

$$G = Q_0 \int_0^t \frac{1}{4\pi D(t-\tau)} e^{-r^2/(4D(t-\tau))-\beta(t-\tau)}\, d\tau .$$

With the new variable

$$\theta = t - \tau ,$$

we obtain

$$G = \frac{Q_0}{4\pi D} \int_0^t e^{-(r^2/4D\theta)-\beta\theta} \frac{d\theta}{\theta} .$$

The Green's function which corresponds to the stationary problem can be obtained from the preceding formula by passage to the limit $t \to \infty$:

$$\bar{G} = \lim_{t\to\infty} G = \frac{Q_0}{4\pi D} \int_0^\infty e^{-(r^2/4D\theta)-\beta\theta} \frac{d\theta}{\theta} .$$

By means of the substitution

$$0 = Ce^{\xi},$$

where C is a fixed constant, we bring this integral to the form

$$\bar{G} = \frac{Q_0}{4\pi D}\int_{-\infty}^{\infty} e^{-[(r^2/4DC)e^{-\xi}+\beta Ce^{\xi}]}\,d\xi .$$

If we require the equation

$$\frac{r^2}{4DC} = \beta C$$

to be satisfied, then we find

$$C = \frac{r}{2\sqrt{\beta D}} \quad \text{and} \quad \frac{r^2}{4DC} = \beta C = \frac{r}{2}\sqrt{\frac{\beta}{D}} = \frac{\kappa r}{2} \quad \left(\kappa^2 = \frac{\beta}{D}\right).$$

From this, it follows that the stationary Green's function has the form

$$\bar{G} = \frac{Q_0}{4\pi D}\int_{-\infty}^{\infty} e^{-\kappa r\cosh\xi}\,d\xi = \frac{Q_0}{2\pi D}\int_{0}^{\infty} e^{-\kappa r\cosh\xi}\,d\xi = \frac{Q_0}{2\pi D}K_0(\kappa r) .$$

Therefore, the problem considered here leads to the integral representation of the function $K_0(x)$.

4. Integral representations and asymptotic representations of Bessel functions

1. *Integral representations for the Bessel functions of integral order.* Now we consider the time periodic solutions of the wave equation

$$u_{xx} + u_{yy} = \frac{1}{a^2}u_{tt} .$$

If we set

$$u(x, y, t) = v(x, y)e^{i\omega t},$$

we obtain for the vibration amplitude $v(x, y)$ the vibration equation

$$v_{xx}v_{yy} + k^2 v = 0 , \qquad \left(k = \frac{\omega}{a}\right). \tag{A-2.95}$$

This has solutions of the form

$$v = e^{\mp ikx} \quad \text{and} \quad v = e^{\mp iky}, \tag{A-2.96}$$

which correspond to plane waves

$$u = e^{i(\omega t \mp kx)} \quad \text{and} \quad u = e^{i(\omega t \mp ky)} .$$

Depending on the choice of positive or negative sign, the plane wave propagates in the positive or negative direction of the axis. In the following discussion we shall omit the time factor $e^{i\omega t}$ and regard the functions (A-2.96) as plane waves.

A plane wave, which propagates in the direction $\boldsymbol{s}$, obviously has the form

$$v = e^{-ik\boldsymbol{s}\boldsymbol{r}} = e^{-ik(x\cos\alpha + y\sin\alpha)} .$$

Here α denotes the angle between the direction $\boldsymbol{s}$ and the x-axis.

If we introduce polar coordinates

$$x = r\cos\varphi\ , \qquad y = r\sin\varphi\ ,$$

then

$$v = e^{-ikr\cos(\varphi-\alpha)}\ .$$

The case $\alpha = 0$, therefore, corresponds to a plane wave which propagates in the direction of the positive x-axis. The case $\alpha = \pi/2$ corresponds to a wave propagating in the direction of the positive y-axis.

Now we expand the plane wave propagating along the y-axis

$$v = e^{ikr\sin\varphi} \tag{A-2.97}$$

in a Fourier series with respect to the variable φ:

$$v = \sum_{n=-\infty}^{\infty} A_n(\rho)e^{in\varphi}\ , \qquad (\rho = kr)\ , \tag{A-2.98}$$

where

$$A_n(\rho) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{i\rho\sin\varphi - in\varphi}\,d\varphi\ . \tag{A-2.99}$$

Then we find

$$A_n(\rho) = J_n(\rho)\ . \tag{A-2.100}$$

In order to prove this, we first have to show that $A_n(\rho)$ satisfies Bessel's equation (see Section A-2.1). We write this in the form

$$\rho^2\frac{d^2y}{d\rho^2} + \rho\frac{dy}{d\rho} + (\rho^2 - n^2)y = \boldsymbol{L}(y) = 0\ .$$

Then we have

$$\boldsymbol{L}(A_n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-i\rho\sin\varphi + in\varphi}[-\rho^2\sin^2\varphi - i\rho\sin\varphi + \rho^2 - n^2]\,d\varphi\ , \tag{A-2.101}$$

and after integrating by parts twice, we obtain

$$\begin{aligned} n^2\int_{-\pi}^{\pi} e^{-i\rho\sin\varphi+in\varphi}\,d\varphi &= n\rho\int_{-\pi}^{\pi} e^{-i\rho\sin\varphi+in\varphi}\cos\varphi\,d\varphi \\ &= -\frac{\rho}{i}\int_{-\pi}^{\pi} e^{-i\rho\sin\varphi+in\varphi}[-i\rho\cos^2\varphi - \sin\varphi]\,d\varphi \\ &= \int_{-\pi}^{\pi} e^{-i\rho\sin\varphi+in\varphi}[\rho^2 - \rho^2\sin^2\varphi - i\rho\sin\varphi]\,d\varphi\ . \end{aligned} \tag{A-2.102}$$

Here we are able to drop those terms which do not occur under the integral, since they are equal to zero because of periodicity. If we put (A-2.102) into formula (A-2.101), we obtain

$$\boldsymbol{L}(A_n) = 0\ . \tag{A-2.103}$$

Further, from Eq. (A-2.99) it can be seen that the function $A_n(\rho)$ for $\rho = 0$ remains bounded. From this and from Eq. (A-2.103), it follows that $A_n(\rho)$ is proportional to $J_n(\rho)$:

$$A_n(\rho) = C_n J_n(\rho)\ . \tag{A-2.104}$$

For the determination of the factor C_n, we compare both sides of formula (A-2.104) for $\rho = 0$. Since $J_n(\rho)$ for $\rho = 0$ possesses a zero of the nth order, by n times differentiation of $J_n(\rho)$ we find (see Vol. I, Appendix, p. 365)

$$\left[\frac{d^n}{d\rho^n} J_n(\rho)\right]_{\rho=0} = \frac{1}{2^n} . \tag{A-2.105}$$

On the other hand, by differentiation of A_n from Eq. (A-2.99) we obtain

$$\left[\frac{d^k}{d\rho^k} A_n(\rho)\right]_{\rho=0} = \frac{1}{2\pi}(-i)^k \int_{-\pi}^{\pi} e^{in\varphi} \sin^k \varphi \, d\varphi = \frac{(-i)^k}{2\pi \cdot 2^k i^k} \int_{-\pi}^{\pi} e^{in\varphi}(e^{i\varphi} - e^{-i\varphi})^k \, a\varphi .$$

Since

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{in\varphi} e^{ik\varphi} \, d\varphi = \begin{cases} 0 & \text{for } k \neq -n , \\ 1 & \text{for } k = -n , \end{cases}$$

then

$$\left[\frac{d^k}{d\rho^k} A_n(\rho)\right]_{\rho=0} = \begin{cases} 0 & \text{for } k < n , \\ \dfrac{(-1)^n}{2^n} \cdot \dfrac{1}{2\pi} \displaystyle\int_{-\pi}^{\pi} e^{in\varphi}(-e^{-i\varphi})^n \, d\varphi = \dfrac{1}{2n} & \text{for } k = n . \end{cases} \tag{A-2.106}$$

If we compare Eqs. (A-2.105) and (A-2.106), then it follows that

$$C_n = 1 .$$

Therefore, for Bessel functions of the first kind with integral order n we have the integral representation

$$J_n(\rho) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-i\rho \sin\varphi - in\varphi} \, d\varphi . \tag{A-2.107}$$

For the plane waves (A-2.97) the expansion

$$e^{-i\rho \sin\varphi} = \sum_{n=-\infty}^{\infty} J_n(\rho) e^{-in\varphi}$$

is valid. In particular, it follows that

$$\cos(\rho \sin\varphi) = J_0(\rho) + 2J_2(\rho)\cos 2\varphi + 2J_4(\rho)\cos 4\varphi + \cdots ,$$
$$\sin(\rho \sin\varphi) = 2J_1(\rho)\sin\varphi + 2J_3(\rho)\sin 3\varphi + \cdots .$$

We shall now give still another integral formula for $J_n(\rho)$. Here we set $\varphi = \psi - \pi/2$. Therefore, from Eq. (A-2.107) we obtain

$$J_n(\rho) = \frac{(-i)^n}{2\pi}\int_{-\pi}^{\pi} e^{i\rho\cos\psi - in\psi} \, d\psi . \tag{A-2.107'}$$

Because of the periodicity of the integrand in Eqs. (A-2.107) and (A-2.107'), the integration over an arbitrary interval of length 2π can be carried out. Formula (A-2.107') can also be derived from the series representation

$$e^{i\rho\cos\varphi} = \sum_{n=-\infty}^{\infty} i^n J_n(\rho) e^{-in\varphi} .$$

Using formulas (A-2.107) and (A-2.107') once more, we write the two integral formulas for $J_0(\rho)$ separately:

$$J_0(\rho) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-i\rho \sin\varphi}\, d\varphi\,, \tag{A-2.108}$$

$$J_0(\rho) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{i\rho\cos\varphi}\, d\varphi\,. \tag{A-2.108'}$$

Let us still note that the integral representation (A-2.107) and (A-2.107′) have been derived only for functions of integer order. If the index ν is not an integer, then we have

$$J_\nu(\rho) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-i\rho\sin\varphi + i\nu\varphi}\, d\varphi - \frac{\sin\nu\pi}{\pi}\int_0^\infty e^{-\rho\sinh\xi - \nu\xi}\, d\xi\,. \tag{A-2.109}$$

This representation is an analog of formula (A-2.107) [see formula (A-2.148)].

2. *Asymptotic representations of Bessel functions of the first kind.* Before we go into the investigation of the asymptotic behavior of Bessel functions, we shall prove the following lemma:

Lemma. A function $Q(\rho)$ represented by an integral of the type

$$Q(\rho) = \frac{1}{\pi}\int_0^1 \frac{e^{\pm i\rho\xi}}{\sqrt{1-\xi}} f(\xi)\, d\xi\,, \tag{A-2.110}$$

where $f(\xi)$ is a continuous and twice differentiable function, can be defined for sufficiently large values of ρ in the form

$$Q(\rho) = \frac{e^{\pm i(\rho - \pi/4)}}{\sqrt{\pi\rho}} f(1)\left[1 + O\left(\frac{1}{\rho}\right)\right]. \tag{A-2.111}$$

Thus $O(1/\rho) \to 0$ for $\rho \to \infty$.

Proof. We carry out the proof for the case in which the plus sign remains in the exponents.

The integrand in Eq. (A-2.110) for $\xi = 1$ is singular. For this reason we introduce the new variable

$$\eta = 1 - \xi\,, \qquad (\xi = 1 - \eta)\,.$$

With it we obtain

$$Q(\rho) = \frac{e^{i\rho}}{\pi}\int_0^1 e^{-i\rho\eta}\frac{f(1-\eta)}{\sqrt{\eta}}\, d\eta\,. \tag{A-2.112}$$

The second factor of the intergrand can be written in the form

$$\frac{f(1-\eta)}{\sqrt{\eta}} = \frac{f(1)}{\sqrt{\eta}} + g(\eta)\,. \tag{A-2.113}$$

if

$$\begin{aligned} g(\eta) &= \frac{f(1-\eta) - f(1)}{\sqrt{\eta}} \\ &= \frac{f(1-\eta) - f(1)}{\eta}\sqrt{\eta} = g_1(\eta)\sqrt{\eta}\,, \qquad (g(0) = 0)\,. \end{aligned} \tag{A-2.114}$$

Here $g_1(\eta)$ is a continuous and differentiable function of η. In particular, $g'(\eta)$ is absolutely integrable, that is,

$$\int_0^1 |g'(\eta)|\, d\eta < M\,,$$

where M is a fixed constant. If we put Eq. (A-2.113) into Eq. (A-2.112), then we obtain

$$Q(\rho) = \frac{e^{i\rho}}{\pi}\left\{f(1)\int_0^1 \frac{e^{-i\rho\eta}}{\sqrt{\eta}}\, d\eta + \int_0^1 e^{-i\rho\eta}\, g(\eta)\, d\eta\right\} = \frac{e^{i\rho}}{\pi}(q_1 + q_2)\,, \qquad \text{(A-2.115)}$$

where

$$q_1 = f(1)\int_0^1 \frac{e^{-i\rho\eta}}{\sqrt{\eta}}\, d\eta\,, \qquad q_2 = \int_0^1 e^{i\rho\eta}\, g(\eta)\, d\eta\,.$$

For $q_1(\rho)$, the expression

$$q_1(\rho) = f(1)\sqrt{\frac{\pi}{\rho}}\, e^{-i\pi/4}\left[1 + O\left(\frac{1}{\rho}\right)\right] \qquad \text{(A-2.116)}$$

results since

$$\int_0^\rho \frac{e^{-i\xi}}{\sqrt{\xi}}\, d\xi = \int_0^\infty \frac{e^{-i\zeta}}{\sqrt{\zeta}}\, d\xi\left[1 + O\left(\frac{1}{\rho}\right)\right] = \sqrt{\pi}\, e^{-i\pi/4}\left[1 + O\left(\frac{1}{\rho}\right)\right].$$

Therefore we obtain

$$\int_0^\infty \frac{e^{-\alpha\xi}}{\sqrt{\xi}}\, d\xi = \frac{2}{\sqrt{\alpha}}\int_0^\infty e^{-t^2}\, dt = \frac{\sqrt{\pi}}{\sqrt{\alpha}}\,.$$

By extension of this formula to complex values of α, whose real part is non-negative, we find[53]

$$\int_0^\infty \frac{e^{-i\xi}}{\sqrt{\xi}}\, d\xi = \frac{\sqrt{\pi}}{\sqrt{i}} = \sqrt{\pi}\, e^{-i\pi/4}\,, \qquad \left(\text{analog } \int_0^\infty \frac{e^{i\xi}}{\sqrt{\xi}}\, d\xi = \frac{\sqrt{\pi}}{\sqrt{-i}} = \sqrt{\pi}\, e^{i\pi/4}\right).$$

The integral q_2 is transformed by means of partial integration:

$$\begin{aligned} q_2 &= \int_0^1 e^{-i\rho\eta}\, g(\eta)\, d\eta = \frac{1}{-i\rho}\, e^{-i\rho\eta}\, g(\eta)\Big|_0^1 - \frac{1}{-i\rho}\int_0^1 e^{i\rho\eta}\, g'(\eta)\, d\eta \\ &= \frac{1}{\sqrt{\rho}}\, O\left(\frac{1}{\rho}\right). \end{aligned} \qquad \text{(A-2.117)}$$

After substituting expressions (A-2.116) and (A-2.117) into formula (A-2.115), we obtain

$$Q(\rho) = \frac{e^{i(\rho - \pi/4)}}{\sqrt{\pi\rho}}\, f(1)\left[1 + O\left(\frac{1}{\rho}\right)\right]. \qquad \text{(A-2.111')}$$

if the plus sign in Eq. (A-2.110) is assumed.

A corresponding formula is obtained when the minus sign is chosen in Eq. (A-2.110):

[53] For details see any good text on complex variables.

$$Q(\rho) = \frac{e^{-i(\rho-\pi/4)}}{\sqrt{\pi\rho}} f(1)\left[1 + O\left(\frac{1}{\rho}\right)\right]. \qquad \text{(A-2.111'')}$$

The lemma is therefore proved.

We shall now use this lemma for the derivation of asymptotic representations for the Bessel functions of the first kind of integer order. With the new variable

$$\cos\phi = \xi$$

and from formula (A-2.107'), we obtain

$$\begin{aligned} J_n(\rho) &= \frac{(-i)^n}{2\pi}\int_{-\pi}^{\pi} e^{i\rho\cos\phi}\cos n\phi\, d\phi = \frac{(-i)^n}{\pi}\int_0^{\pi} e^{i\rho\cos\phi}\cos n\phi\, d\phi \\ &= \frac{(-i)^n}{\pi}\int_{-1}^{1} e^{i\rho\xi}\frac{T_n(\xi)}{\sqrt{1-\xi^2}}\, d\xi\,, \end{aligned} \qquad \text{(A-2.118)}$$

where

$$T_n(\xi) = \cos(n \arccos\xi)\,.$$

Since

$$\cos n\phi = \mathrm{Re}(\cos\phi + i\sin\phi)^n$$

is a polynomial of the nth degree in $\cos\phi$, on the basis of DeMoivre's formula, $T_n(\xi)$ also represents a polynomial of the nth degree with respect to $\xi = \cos\phi$. The polynomial $T_n(\xi)$ is called the Tschebyscheff polynomial.

It satisfies the relation

$$\begin{aligned} T_n(-\xi) &= \cos(n \arccos(-\xi)) = \cos(n\pi - n\arccos\xi) \\ &= \cos n\pi\cos(n\arccos\xi) + \sin n\pi\sin(n\arccos\xi) = (-1)^n T_n(\xi)\,; \end{aligned}$$

therefore

$$T_n(-\xi) = (-1)^n T_n(\xi)$$

holds.

Now if the integral (A-2.118) is represented as the sum of integrals from -1 to 0 and from 0 to $+1$, bearing in mind the last named property of the Tschebyscheff polynomials, we obtain

$$J_n(\rho) = \frac{(-i)^n}{\pi}\int_0^1 \frac{e^{i\rho\xi}}{\sqrt{1-\xi}} f(\xi)\, d\xi + \frac{i^n}{\pi}\int_0^1 \frac{e^{-i\rho\xi}}{\sqrt{1-\xi}} f(\xi)\, d\xi\,, \qquad \text{(A-2.118')}$$

where $f(\xi) = T_n(\xi)/\sqrt{1+\xi}$ is a continuous and arbitrarily often differentiable function with $f(1) = 1/\sqrt{2}$.

By application of the last proved lemma to each of the integrals of (A-2.118') we find from formulas (A-2.111') and (A-2.111'') that

$$\begin{aligned} J_n(\rho) &= \frac{1}{\sqrt{2\pi\rho}}\left[e^{i(\rho-\pi/4-n\pi/2)} + e^{-i(\rho-\pi/4-n\pi/2)}\right]\left[1 + O\left(\frac{1}{\rho}\right)\right] \\ &= \sqrt{\frac{2}{\pi\rho}}\cos\left[\rho - \frac{\pi}{2}(n+\tfrac{1}{2})\right]\left[1 + O\left(\frac{1}{\rho}\right)\right]. \end{aligned}$$

Therefore, for Bessel functions of the first kind with integer order, there holds for large values of the argument the asymptotic representation

$$J_n(\rho) = \sqrt{\frac{2}{\pi\rho}} \cos\left[\rho - \frac{\pi}{2}(n + \tfrac{1}{2})\right]\left[1 + O\left(\frac{1}{\rho}\right)\right]. \qquad \text{(A-2.119)}$$

In particular, for $n = 0$,

$$J_0(\rho) = \sqrt{\frac{2}{\pi\rho}} \cos\left(\rho - \frac{\pi}{4}\right)\left[1 + O\left(\frac{1}{\rho}\right)\right], \qquad \text{(A-2.119')}$$

and for $n = 1$,

$$J_1(\rho) = -\sqrt{\frac{2}{\pi\rho}} \cos\left(\rho + \frac{\pi}{4}\right)\left[1 + O\left(\frac{1}{\rho}\right)\right]. \qquad \text{(A-2.119'')}$$

In Section A-2.6 the validity of the following asymptotic representations for the Hankel functions will be shown:

$$H_n^{(1)}(\rho) = \sqrt{\frac{2}{\pi\rho}}\, e^{i[\rho - \pi/2(n+1/2)]}\left[1 + O\left(\frac{1}{\rho}\right)\right], \qquad \text{(A-2.120)}$$

$$H_n^{(2)}(\rho) = \sqrt{\frac{2}{\pi\rho}}\, e^{-i[\rho - \pi/2(n+1/2)]}\left[1 + O\left(\frac{1}{\rho}\right)\right]. \qquad \text{(A-2.121)}$$

From Eqs. (A-2.120) and (A-2.121) with

$$H_n^{(1)}(\rho) = J_n(\rho) + iN_n(\rho) \qquad \text{(A-2.65)}$$

and

$$H_n^{(2)}(\rho) = J_n(\rho) - iN_n(\rho)\,, \qquad \text{(A-2.66)}$$

the asymptotic representation

$$N_n(\rho) = \sqrt{\frac{2}{\pi\rho}} \sin\left[\rho - \frac{\pi}{2}(n + \tfrac{1}{2})\right]\left[1 + O\left(\frac{1}{\rho}\right)\right] \qquad \text{(A-2.122)}$$

of the Neumann functions results.

Formulas (A-2.65) and (A-2.66) together with the asymptotic representations for $H_n^{(1)}(\rho)$, $H_n^{(2)}(\rho)$, $J_n(\rho)$, and $N_n(\rho)$ are, with regard to their structure, similar to the Euler formulas for the trigonometric functions

$$e^{ix} = \cos x + i \sin x\,, \qquad e^{-ix} = \cos x - i \sin x\,.$$

5. The Fourier-Bessel integral and some integrals which contain Bessel functions

1. *Fourier-Bessel integral.* We shall now analyze a given function in an integral by means of Bessel functions. As is known, the Fourier integral of a function $f(x)$ and, correspondingly, a function $f(x, y)$ of two variables has the form

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\mu \int_{-\infty}^{\infty} f(\xi) e^{i\mu(x-\xi)}\, d\xi\,, \qquad \text{(A-2.123)}$$

$$f(x, y) = \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} d\mu\, d\mu' \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\xi, \eta) e^{i\mu(x-\xi) - i\mu'(y-\eta)}\, d\xi\, d\eta\,. \qquad \text{(A-2.124)}$$

If we introduce polar coordinates by means of the expressions

$$x = r\cos\varphi\,, \qquad \xi = \rho\cos\psi\,, \qquad \mu = \lambda\cos\xi\,,$$
$$y = r\sin\varphi\,, \qquad \eta = \rho\sin\psi\,, \qquad \mu' = \lambda\sin\xi\,,$$

then we obtain

$$d\xi\,d\eta = \rho\,d\rho\,d\psi\,, \qquad d\mu\,d\mu' = \lambda\,d\lambda\,d\xi\,,$$
$$\mu x + \mu' y = \lambda r\cos(\xi - \varphi)\,,$$
$$\mu\xi + \mu'\eta = \lambda\rho\cos(\psi - \xi)\,.$$

In the following discussion we assume that $f(x, y)$ has the form

$$f(x, y) = f(r)e^{in\varphi}\,, \tag{A-2.125}$$

where n is an integer. Then by the above transformations we obtain the Fourier integrals (A-2.124) in polar coordinates

$$f(r)e^{in\varphi} = \int_0^\infty\int_0^\infty f(\rho)\,\rho\,d\rho\,\lambda\,d\lambda\cdot\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{i\lambda r\cos(\xi-\varphi)+in(\xi-\varphi)}\,d\xi\cdot e^{in\varphi}$$
$$\cdot\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-i\lambda\rho\cos(\psi-\xi)+in(\psi-\xi)}\,d\psi\,. \tag{A-2.126}$$

Now to treat these expressions further, we use the formulas

$$J_n(z) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{iz\cos\xi+in\xi}e^{-i\pi n/2}\,d\xi\,, \tag{A-2.127}$$

$$J_n(z) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-iz\cos\xi'+in\xi'}e^{i\pi n/2}\,d\xi'\,, \qquad (\xi = \pi + \xi')\,. \tag{A-2.128}$$

Since the integrands in Eqs. (A-2.127) and (A-2.128) are periodic functions of ξ and ξ', on this basis the integration over an arbitrary interval of length 2π can be carried out, and we can write

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{iz\cos(\xi-\xi_0)+in(\xi-\xi_0)}\,d\xi = J_n(z)e^{i\pi n/2}\,, \tag{A-2.129}$$

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-iz\cos(\xi'-\xi_0')+in(\xi'-\xi_0')}\,d\xi' = J_n(z)e^{-i\pi n/2}\,, \tag{A-2.130}$$

where ξ_0 and ξ_0' are arbitrary numbers. By introduction of Eqs. (A-2.129) and (A-2.130) in (A-2.126) and by division of both sides by $e^{in\varphi}$, we then obtain the Fourier-Bessel integral

$$f(r) = \int_0^\infty\int_0^\infty f(\rho)J_n(\lambda\rho)J_n(\lambda r)\lambda\,d\lambda\,\rho\,d\rho \tag{A-2.131}$$

or

$$f(r) = \int_0^\infty \varphi(\lambda)J_n(\lambda r)\lambda\,d\lambda\,,$$

where

$$\varphi(\lambda) = \int_0^\infty f(\rho) J_n(\lambda\rho)\rho\, d\rho .$$

The representation as a Fourier Bessel integral is possible when the function $f(r)$ defined in the interval $(0, \infty)$ satisfies the following conditions:

(a) $f(r)$ is continuous in the interval $(0, \infty)$.

(b) $f(r)$ in each finite interval possesses only finitely many maxima and minima.

(c) The integral

$$\int_0^\infty \rho|f(\rho)|\, d\rho$$

exists.

We shall not go into the proof of this assertion here.

2. *Some integrals whose integrands contain Bessel functions.* For different applications, definite integrals occur which contain Bessel functions.

An oft occurring integral of this type is the integral

$$B_1 = \int_0^\infty e^{-z\lambda} J_0(\rho\lambda)\, d\lambda = \frac{1}{\sqrt{\rho^2 + z^2}} , \qquad (z > 0) . \qquad \text{(A-2.132)}$$

For a proof of this formula we replace J_0 by the corresponding integral representation (A-2.108). If we interchange the order of integration, we obtain

$$\begin{aligned} B_1 &= \int_0^\infty e^{-z\lambda} J_0(\rho\lambda)\, d\lambda = \frac{1}{2\pi}\int_0^\infty e^{-z\lambda}\, d\lambda \int_{-\pi}^{\pi} e^{i\rho\lambda\sin\varphi}\, d\varphi \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} d\varphi \int_0^\infty e^{-(z - i\rho\sin\varphi)\lambda}\, d\lambda = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{d\varphi}{z + i\rho\sin\varphi} \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{z\, d\varphi}{z^2 + \rho^2\sin^2\varphi} - \frac{i}{2\pi}\int_{-\pi}^{\pi} \frac{\rho\sin\varphi\, d\varphi}{z^2 + \rho^2\sin^2\varphi} = \frac{1}{\pi}\int_0^{\pi} \frac{z\, d\varphi}{z^2 + \rho^2\sin^2\varphi} , \end{aligned}$$

since

$$\int_{-\pi}^{\pi} \frac{\sin\varphi\, d\varphi}{z^2 + \rho^2\sin^2\varphi} = 0$$

because of the odd integrands.

Next, if we set $\tan\varphi = \xi$ and then set $(\sqrt{(z^2 + \rho^2)/z^2})\xi = \eta$, we obtain

$$\begin{aligned} B_1 &= \frac{1}{\pi}\int_0^{\pi} \frac{z\, d\varphi}{z^2 + \rho^2\sin^2\varphi} = \frac{2z}{\pi}\int_0^{\pi/2} \frac{d\varphi}{z^2 + \rho^2\sin^2\varphi} \\ &= \frac{2z}{\pi}\int_0^\infty \frac{d\xi}{z^2(1 + \xi^2) + \rho^2\xi^2} = \frac{2}{\pi\sqrt{z^2 + \rho^2}}\int_0^\infty \frac{d\eta}{1 + \eta^2} = \frac{1}{\sqrt{z^2 + \rho^2}} , \end{aligned}$$

whereby the validity of formula (A 2.132) is proved. From formula (A 2.132) we immediately find

$$\int_0^\infty J_1(\rho\lambda) e^{-z\lambda}\, d\lambda = \frac{1}{\rho}\left(1 - \frac{z}{\sqrt{z^2 + \rho^2}}\right) . \qquad \text{(A-2.133)}$$

If in Eqs. (A-2.132) and (A 2.133) we set $z = ia$, then by splitting the resulting equations into real and imaginary parts, we obtain the expressions

$$\left.\begin{aligned}
&\int_0^\infty J_0(\rho\lambda)\cos a\lambda\, d\lambda = \frac{1}{\sqrt{\rho^2 - a^2}}\,,\\
&\int_0^\infty J_0(\rho\lambda)\sin a\lambda\, d\lambda = 0\,,\\
&\int_0^\infty J_1(\rho\lambda)\cos a\lambda\, d\lambda = \frac{1}{\rho}\,,\\
&\int_0^\infty J_1(\rho\lambda)\sin a\lambda\, d\lambda = \frac{1}{\rho\sqrt{\rho^2 - a^2}}\,,
\end{aligned}\right\} \text{ for } \rho > a\,; \qquad \text{(A-2.134)}$$

$$\left.\begin{aligned}
&\int_0^\infty J_0(\rho\lambda)\cos a\lambda\, d\lambda = 0\,,\\
&\int_0^\infty J_0(\rho\lambda)\sin a\lambda\, d\lambda = \frac{1}{\sqrt{a^2 - \rho^2}}\,,
\end{aligned}\right\} \text{ for } a > \rho\,; \qquad \text{(A-2.135)}$$

$$\left.\begin{aligned}
&\int_0^\infty J_1(\rho\lambda)\cos a\lambda\, d\lambda = \frac{1}{\rho}\left(1 - \frac{a}{\sqrt{a^2 - \rho^2}}\right),\\
&\int_0^\infty J_1(\rho\lambda)\sin a\lambda\, d\lambda = 0\,.
\end{aligned}\right\} \text{ for } a > \rho\,. \qquad \text{(A-2.135')}$$

Now we prove the second integral formula .

$$B_2 = \int_0^\infty J_\nu(\lambda\rho)e^{-t\lambda^2}\lambda^{\nu+1}\, d\lambda = \frac{1}{2t}\left(\frac{\rho}{2t}\right)^\nu e^{-\rho^2/4t}\,. \qquad \text{(A-2.136)}$$

In this formula we replace J_ν by the corresponding power series and integrate term by term ($t > 0$):

$$B_2 = \sum_{k=0}^{\infty} \frac{(-1)^k}{\Gamma(k+1)\Gamma(k+\nu+1)}\left(\frac{\rho}{2}\right)^{2k+\nu}\int_0^\infty \lambda^{2k+2\nu+1}e^{-t\lambda^2}\, d\lambda\,.$$

We find

$$\int_0^\infty \lambda^{2k+2\nu+1}e^{-t\lambda^2}\, d\lambda = \frac{1}{2t^{k+\nu+1}}\int_0^\infty e^{-\xi}\xi^{k+\nu}\, d\xi = \frac{1}{2t^{k+\nu+1}}\Gamma(k+\nu+1)\,.$$

From this we obtain

$$B_2 = \frac{1}{2t}\left(\frac{\rho}{2t}\right)^\nu \sum_{k=0}^{\infty}\frac{(-1)^k}{k!}\left(\frac{\rho^2}{4t}\right)^k = \frac{1}{2t}\left(\frac{\rho}{2t}\right)^k e^{-\rho^2/4t}\,,$$

which was to be proved.

Thus we have proved that the integral B_1 can be calculated analogously when the Bessel function is expanded in its power series and thereafter integrated term by term.

Now we consider the integral

$$C = \int_0^\infty J_0(\lambda\rho)\frac{e^{-\sqrt{\lambda^2-k^2}\,|z|}}{\sqrt{\lambda^2 - k^2}}\lambda\, d\lambda\,. \qquad \text{(A-2.137)}$$

As we can easily prove, it satisfies the equation

$$\Delta v + k^2 v = 0\,, \qquad \left(\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right).$$

The function

$$v_0 = \frac{e^{ikr}}{r}, \qquad (r = \sqrt{\rho^2 + z^2})$$

likewise satisfies the vibration equation

$$\Delta v_0 + k^2 v_0 = \frac{1}{r}\frac{\partial^2}{\partial r^2}(rv_0) + k^2 v_0 = -k^2\frac{e^{ikr}}{r} + k^2 v_0 = 0 .$$

The representation of the function $v_0(\rho) = e^{ik\rho}/\rho$ as a Fourier-Bessel integral reads

$$\frac{e^{ik\rho}}{\rho} = \int_0^\infty F(\lambda) J_0(\rho\lambda)\lambda\, d\lambda , \tag{A-2.138}$$

where

$$F(\lambda) = \int_0^\infty e^{ik\rho} J_0(\lambda\rho)\, d\rho . \tag{A-2.139}$$

For the calculation of $F(\lambda)$ we use formula (A-2.134):

$$F(\lambda) = \int_0^\infty J_0(\lambda\rho)(\cos k\rho + i \sin k\rho)\, d\rho = \begin{cases} \dfrac{1}{\sqrt{\lambda^2 - k^2}}, & \text{for } \lambda > k ; \\ \dfrac{i}{\sqrt{k^2 - \lambda^2}} = \dfrac{1}{\sqrt{\lambda^2 - k^2}}, & \text{for } k > \lambda . \end{cases}$$

Therefore,

$$\frac{e^{ik\rho}}{\rho} = \int_0^\infty J_0(\rho\lambda)\frac{\lambda\, d\lambda}{\sqrt{\lambda^2 - k^2}} , \tag{A-2.140}$$

that is, the function

$$v_0 = \frac{e^{ik\sqrt{\rho^2 + z^2}}}{\sqrt{\rho^2 + z^2}}$$

coincides with the integral $C(\rho, z)$ for $z = 0$. Both functions

$$v_0(\rho, z) \qquad \text{and} \qquad C(\rho, z)$$

are thus solutions of the vibration equation which coincide for $z = 0$. At the point $z = 0$, $\rho = 0$ it has a singularity of the same type.

However, it follows from this that they are identical with each other, that is,

$$\int_0^\infty J_0(\lambda\rho)\frac{e^{-\sqrt{\lambda^2 - k^2}|z|}}{\sqrt{\lambda^2 - k^2}}\lambda\, d\lambda = \frac{e^{ik\sqrt{\rho^2 + z^2}}}{\sqrt{\rho^2 + z^2}} . \tag{A-2.141}$$

The formula obtained was used principally by A. Sommerfeld for physical investigations. For this reason it is often called the Sommerfeld formula.

6. Representation of cylindrical functions by contour integrals

1. *Representation of cylindrical functions by contour integrals.* With the help of contour integrals, the solutions of Bessel's differential equation can often be represented in the form

$$\int_C e^{-ix\sin\varphi - i\nu\varphi}\, d\varphi = \int_C K(x,\varphi)\Phi(\varphi)\, d\varphi\,, \tag{A-2.142}$$

where C is a curve lying in the plane of the complex variable φ and $K(x,\varphi) = e^{-ix\sin\varphi}$, $\Phi(\varphi) = e^{i\nu\varphi}$. A function of the form (A-2.142) can be considered as a superposition of plane waves

$$e^{-i(x\sin\varphi - y\cos\varphi)}\,,$$

which propagate with different angles with respect to the y-axis; $\Phi = e^{i\nu\varphi}$ appears as the amplitude factor of the plane waves.

We shall now settle the question of the convergence of integrals (A-2.142). In Figure 92 the shaded area represents those parts of the plane of the com-

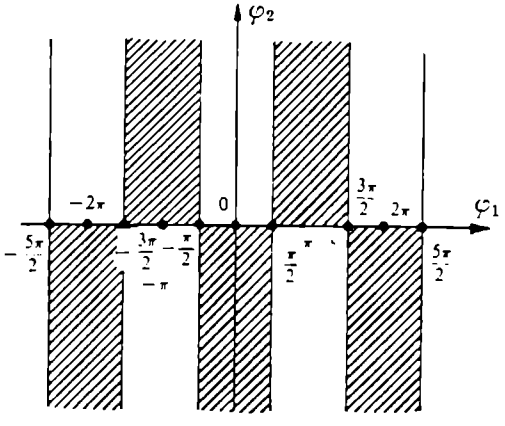


FIG. 92

plex variable $\varphi = \varphi_1 + i\varphi_2$ where the real part of the expression

$$\begin{aligned} ix\sin\varphi &= xi\sin(\varphi_1 + i\varphi_2) \\ &= x(i\sin\varphi_1\cosh\varphi_2 - \cos\varphi_1\sinh\varphi_2) \end{aligned} \tag{A-2.143}$$

is positive (for $x > 0$).

Hankel functions of the first and second kind are usually defined by the integrals

$$H_\nu^{(1)}(x) = -\frac{1}{\pi}\int_{C_1} e^{-ix\sin\varphi + i\nu\varphi}\, d\varphi\,, \tag{A-2.144}$$

$$H_\nu^{(2)}(x) = -\frac{1}{\pi}\int_{C_2} e^{-ix\sin\varphi - i\nu\varphi}\, d\varphi\,. \tag{A-2.145}$$

Here curve C_1 consists of the lines $(-i\infty, 0)$, $(0, -\pi)$, $(-\pi, -\pi + i\infty)$ and C_2 of the lines $(\pi + i\infty, \pi)$, $(\pi, 0)$, $(0, -i\infty)$ (Figure 93, paths I and II). Obviously these integrals converge along C_1 and C_2 for $x > 0$.

Now we shall prove that the functions (A-2.144) and A-2.145) satisfy the Bessel equation

$$L(y) = x_2 y'' + xy' + (x^2 - \nu^2)y = 0\,.$$

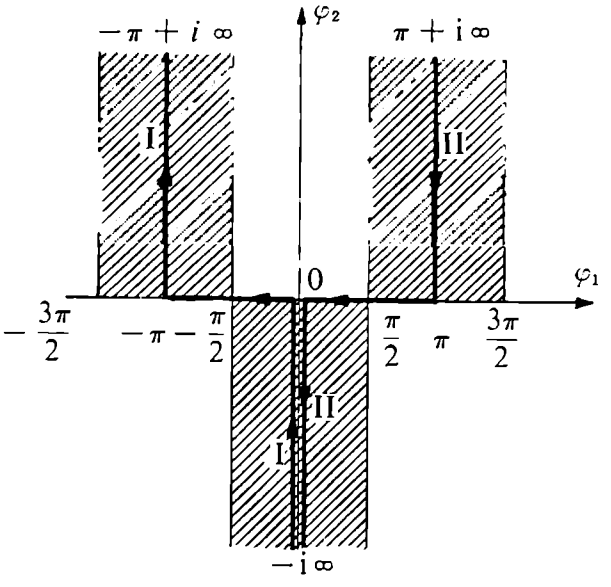


FIG. 93

If we consider that the function

$$K(x, \varphi) = e^{-ix\sin\varphi}$$

satisfies the equation

$$x^2K_{xx} + xK_x + x^2K + K_{\varphi\varphi} = 0\,,$$

then after a twice repeated partial integration we obtain

$$L[H_\nu^{(1)}(x)] = -\frac{1}{\pi}\int_{C_1} L(K)e^{i\nu\varphi}\,d\varphi = \frac{1}{\pi}\int_{C_1}[K_{\varphi\varphi} + \nu^2K]\Phi\,d\varphi$$
$$= \frac{1}{\pi}\int_{C_1}(\Phi'' + \nu^2\Phi)K\,d\varphi + \frac{1}{\pi}\int_{C_1}\frac{\partial}{\partial\varphi}(K_\varphi\Phi - K\Phi')\,d\varphi\,.$$

The first integral on the right side is equal to zero because of the choice of the function Φ, while the second integral can be brought to a form which is likewise equal to zero for $x > 0$ because of the special choice of the path of integration. Consequently, $H_\nu^{(1)}(x)$ and $H_\nu^{(2)}(x)$ are solutions of the Bessel equation.

Analogous considerations show that the above integral along an arbitrary curve C satisfies the Bessel differential equation if the infinite branch of C lies in the shaded area of Figure 93. In the following discussion we shall show that the integral extended over the curve $C_0 = C_2 + C_1$:

$$J_\nu(x) = \frac{1}{2}[H_\nu^{(1)}(x) + H_\nu^{(2)}(x)] = -\frac{1}{2\pi}\int_{C_0} e^{-ix\sin\varphi + i\nu\varphi}\,d\varphi \qquad \text{(A-2.146)}$$

is the Bessel function of the first kind of νth order.

For integer values $\nu = n$, the integral over the infinite branch of paths I and II (see Figure 93) cancel because of the periodicity of the oppositely directed integrands, and there remains, therefore, the integral

$$J_n(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-ix\sin\varphi + in\varphi}\,d\varphi\,. \tag{A-2.147}$$

But this coincides with the representation of the Bessel functions of the first kind of integer order, as given in Section A-2.3.

For noninteger ν-values, the formula derived from Eq. (A-2.146)

$$J_\nu(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-ix\sin\varphi + i\nu\varphi}\,d\varphi - \frac{\sin\nu\pi}{\pi}\int_0^\infty e^{-x\sinh\xi - \nu\xi}\,d\xi \tag{A-2.148}$$

holds.

Integral (A-2.148) remains finite, as we can easily see, for $\nu > 0$ at the point $x = 0$. From this it follows that the function defined by Eq. (A-2.148) must agree to within a factor with the function defined by formula (A-2.77). If we compare the coefficients in the corresponding power series expansion in x, then we recognize that both functions are identical.[54]

We shall now show that for the Hankel functions $H_\nu^{(k)}(x)$ $(k = 1, 2)$, recursion formulas are valid which are analogous to formulas (A-2.21) for the functions $J_\nu(x)$.

If we consider that in formulas (A-2.144) and (A-2.145), the integration paths C_1 and C_2 are not dependent on ν, then we can write

$$H_{\nu+1}^{(k)} + H_{\nu-1}^{(k)} = -\frac{2}{\pi}\int_{C_k} e^{-ix\sin\varphi + i\nu\varphi}\cos\varphi\,d\varphi = \frac{2}{\pi ix}\int_{C_k}\frac{\partial}{\partial\varphi}(e^{-ix\sin\varphi})e^{i\nu\varphi}\,d\varphi\,,$$

$$H_{\nu+1}^{(k)} - H_{\nu-1}^{(k)} = -\frac{2i}{\pi}\int_{C_k} e^{-ix\sin\varphi + i\nu\varphi}\sin\varphi\,d\varphi$$

$$= 2\frac{\partial}{\partial x}\frac{1}{\pi}\int_{C_k} e^{-ix\sin\varphi + i\nu\varphi}\,d\varphi = -2\frac{dH_\nu^{(k)}}{dx}\,, \qquad (k = 1, 2)\,.$$

Partial integration in the first equation then gives for the Hankel functions the recursion formulas

$$H_{\nu+1}^{(1)} + H_{\nu-1}^{(1)} = \frac{2\nu}{x}H_\nu^{(1)}\,, \qquad H_{\nu+1}^{(1)} - H_{\nu-1}^{(1)} = -2\frac{dH_\nu^{(1)}}{dx} \tag{A-2.149}$$

and, analogously,

$$H_{\nu+1}^{(2)} + H_{\nu-1}^{(2)} = \frac{2\nu}{x}H_\nu^{(2)}\,, \qquad H_{\nu+1}^{(2)} - H_{\nu-1}^{(2)} = -2\frac{dH_\nu^{(2)}}{dx}\,. \tag{A-2.149'}$$

We particularly note that following from the above are equations

$$\frac{dH_0^{(1)}(x)}{dx} = -H_1^{(1)}(x)\,, \qquad \frac{dH_0^{(2)}(x)}{dx} = -H_1^{(2)}(x)\,. \tag{A-2.150}$$

If $H_\nu^{(1)}(x)$ and $H_\nu^{(2)}(x)$ are considered as functions of the complex variable $x = x_1 + ix_2$, then formulas (A-2.144) and (A-2.145) define these functions in that region of the variable $x = x_1 + ix_2$ in which the real part

$$\mathrm{Re}\,(-ix\sin\varphi) = \mathrm{Re}\,[(-ix_1 + x_2)\sin(\varphi_1 + i\varphi_2)]$$
$$= x_2\sin\varphi_1\cosh\varphi_2 + x_1\cos\varphi_1\sinh\varphi_2$$

[54] See, for example, A. Sommerfeld, *Lectures on Theoretical Physics*, Vol. VI: "Partial Differential Equations in Physics." Academic Press, New York, 1949.

is negative. If $\varphi_1 = 0$ or $\varphi_1 = \pi$, then integral (A-2.144) along C_1 defines the function $H_\nu^{(1)}(x)$ for all those x values for which the real part $\mathrm{Re}(x) = x_1 > 0$.

Now we denote by $C_{1\psi}$ that path (Figure 94) for which the vertical portion

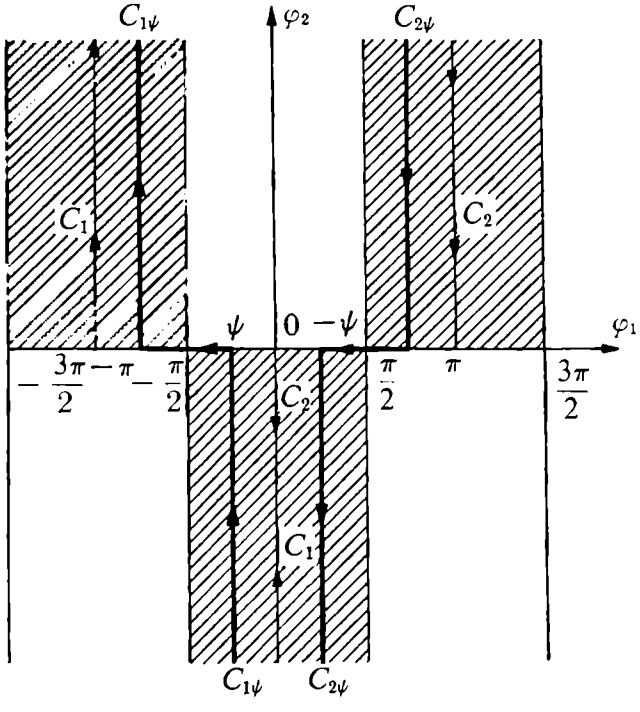


FIG. 94

of the curve C_1 has, instead of $-\pi$ and 0, the abscissa $-\pi - \psi$ and ψ ($\psi < 0$). In particular, $C_{1_0} = C_1$. On the basis of the Cauchy integral theorem, the value of the integral in formula (A-2.144) does not change when the integration path C_1 is replaced by the path $C_{1\psi}$. In the integration along $C_{1\psi}$ the integral (A-2.144) converges for those values $x = x_1 + ix_2$, for which

$$\mathrm{Re}\,(-ix\sin\varphi) = x_2 \sin\varphi_1 \cosh\varphi_2 + x_1 \cos\varphi_1 \sinh\varphi_2 < 0$$

holds for large values of $|\varphi_2|$. Along the lower part of the straight lines $\varphi_1 = \psi$, $\varphi_2 < 0$, this condition is satisfied provided $x_2 \sin\psi - x_1 \cos\psi < 0$. Along the upper part of the straight lines $\varphi = -\pi - \psi$, $\varphi_2 > 0$, the convergence condition is also equivalent to the inequality

$$x_2 \sin\psi - x_1 \cos\psi < 0\,.$$

Consequently, the integral extended over $C_{1\psi}$ converges when the inequality

$$x_2 \sin\psi - x_1 \cos\psi < 0$$

is satisfied (or $x_1 \cos\psi_0 + x_2 \sin\psi_0 > 0$ with $\psi_0 = -\psi$).

If $0 \leqq \psi_0 < \pi/2$, then the integral (A-2.144) over the path $C_{1\psi}$ coincides for real values $x = x_1$; moreover, it defines the function $H_\nu^{(1)}(x)$ in different

regions of the complex x-plane. Consequently, the integral (A-2.144) over curves of the type $C_{1\psi}$ by varying ψ, defines the analytic continuation of $H_\nu^{(1)}(x)$. In particular, if we set

$$\psi_0 = \frac{\pi}{2} \qquad \text{and} \qquad \varphi = -\frac{\pi}{2} + i\eta ,$$

then we obtain the formula

$$H_\nu^{(1)}(x) = -\frac{1}{\pi}\int_{C_{1\psi_0}} e^{-ix\sin\varphi + i\nu\varphi}\, d\varphi = \frac{e^{-\pi\nu i/2}}{\pi i}\int_{-\infty}^{\infty} e^{ix\cosh\eta - \nu\eta}\, d\eta \qquad \text{(A-2.144)}$$

by which $H_\nu^{(1)}(x)$ is defined in the half plane $x_2 = Im(x) > 0$. Hence we obtain for Hankel functions, with pure imaginary argument, the expression

$$H_\nu^{(1)}(ix) = \frac{1}{\pi i} e^{-\nu\pi i/2}\int_{-\infty}^{\infty} e^{-x\cosh\eta - \nu\eta}\, d\eta , \qquad (x > 0) . \qquad \text{(A-2.144''')}$$

The function $K_\nu(x)$ can, according to its definition, be expressed by the corresponding Hankel function in the following way (see Section A-2.3):

$$K_\nu(x) = \frac{\pi}{2} i e^{i\nu\pi/2} H_\nu^{(1)}(ix) .$$

By consideration of formula (A-2.144'''), we therefore obtain the integral representation

$$K_\nu(x) = \tfrac{1}{2}\int_{-\infty}^{\infty} e^{-x\cosh\eta - \nu\eta}\, d\eta .$$

From this it follows that $K_\nu(x)$ is a real function of the argument x. For $\nu = 0$, the formula

$$K_0(x) = \tfrac{1}{2}\int_{-\infty}^{\infty} e^{-x\cosh\eta}\, d\eta = \int_0^{\infty} e^{-x\cosh\eta}\, d\eta$$

has already been derived (see Section A-2.4).

2. *The saddle point method. Asymptotic representations.* The integrand

$$e^{-ix\sin\varphi + i\nu\varphi}$$

in formulas (A-2.144) and (A-2.145), which define the functions $H_\nu^{(1)}(x)$ and $H_\nu^{(2)}(x)$, possesses no singularities in a finite part of the complex φ-plane. For this reason the path of integration C_1, according to the Cauchy integral theorem, can be subjected to arbitrary deformations in a finite region. However, the asymptotes of the branches extending to infinity and the path C_1 itself must still lie in the shaded part of the φ-plane. Correspondingly, the same is valid for the path C_2.

If the path of integration C_1 is chosen as shown in Figure 95, then the integrand at all the points in which $\sin\varphi \neq 0$ approaches zero exponentially as $x \to \infty$, since $Im(\sin\varphi) < 0$. If some part of the paths pass through the unshaded region, then there occurs in the integrands complicated interference phenomena.

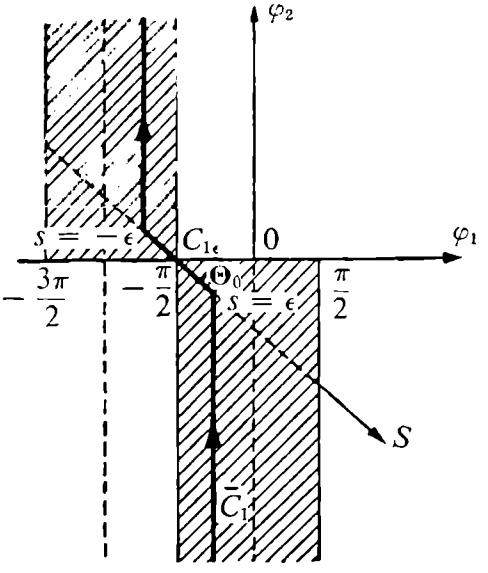


FIG. 95

To understand the asymptotic behavior of $H_\nu^{(1)}(x)$ for large values of x, the path C_1 is suitably selected so that it lies entirely in the shaded region. One such path obviously passes through the point $-\pi/2$ in which the real part

$$\mathrm{Re}\,(-i\sin\varphi) = \cos\varphi_1 \sinh\varphi_2$$

is equal to zero. For $x \to \infty$, the integrand in the neighborhood of this point does not approach zero uniformly, since the principal part of the integral over $\bar{C}_1$ for $x \to \infty$ represents an integral which extends over a small path containing the point $-\pi/2$. On this basis, we must choose the path C_1 so that

$$e^{-ix\sin\varphi}$$

becomes small as rapidly as possible when we move away from the point $\varphi = -\pi/2$. We now consider the path of the function $e^{-ix\sin\varphi}$ in the neighborhood of $\varphi = -\pi/2$.

For this we set

$$\varphi = -\frac{\pi}{2} + se^{i\theta}\,.$$

Then, for small values of s,

$$-i\sin\varphi = i\cos(se^{i\theta}) = i\left(1 - \frac{s^2}{2}e^{2i\theta} + \cdots\right)$$

$$= \frac{s^2}{2}\sin 2\theta + i\left(1 - \frac{s^2}{2}\cos 2\theta\right) + \cdots\,,$$

because the real part of $(s^2/2)\sin 2\theta$ represents a saddle point at the point $s = 0$. In the shaded region this function is negative, in the unshaded region it is positive, and for $s = 0$ ($\varphi = -\pi/2$) it is equal to zero. That direction of traversal through the saddle point which corresponds to the value $Q_0 = -\pi/4$ represents the direction of the steepest slope for the function $(s^2/2)\sin 2\theta$.

From this, it follows that the point $s = 0$ also represents a saddle point for the absolute value of the function $e^{-i \sin \varphi}$, while $\theta_0 = -\pi/4$ corresponds to the direction of the steepest slope.

Now we choose the path $\bar{C}_1$ so that it contains the straight line segment $\bar{C}_{1\varepsilon}$ $(-\varepsilon < s < \varepsilon)$, which passes through the point $s = 0$ with angle $\theta_0 = -\pi/4$, while the branch extending to infinity lies entirely in the shaded region. Since the integrand in formula (A-2.144) decreases exponentially when the argument moves away from the point $s = 0$ $(\varphi = -\pi/2)$, then

$$H_\nu^{(1)}(x) = -\frac{1}{\pi}\int_{C_\varepsilon} e^{-ix\sin\varphi + i\nu\varphi}\, d\varphi \approx \frac{1}{\pi}\int_{-\varepsilon}^{\varepsilon} e^{x(-(s^2/2)+i)-i\nu\pi/2}\, ds \cdot e^{-i\pi/4}$$

up to an exponentially decreasing summand. This is because along the path $\bar{C}_{1_R}$,

$$\varphi = -\frac{\pi}{2} + se^{-i\pi/4}, \qquad d\varphi = e^{-i\pi/4}\, ds,$$

$$-i\sin\varphi \approx -\frac{s^2}{2} + i. \qquad e^{i\nu\varphi} \approx e^{-i\nu\pi/2}$$

holds, where s varies from ε to $-\varepsilon$. With the expression

$$\xi = s\sqrt{\frac{x}{2}}, \qquad d\xi = \sqrt{\frac{x}{2}}\, ds,$$

we thus obtain

$$H_\nu^{(1)}(x) = \frac{1}{\pi}\sqrt{\frac{2}{x}}\, e^{i(x-(\pi/2)\nu-\pi/4)}\int_{-\varepsilon\sqrt{x/2}}^{\varepsilon\sqrt{x/2}} e^{-\xi^2}\, d\xi. \qquad \text{(A-2.151)}$$

If $x \to \infty$, then we obtain[55]

$$\int_{-\varepsilon\sqrt{x/2}}^{\varepsilon\sqrt{x/2}} e^{-\xi^2}\, d\xi \to \int_{-\infty}^{\infty} e^{-\xi^2}\, d\xi = \sqrt{\pi}.$$

From this result and from Eq. (A-2.151), for sufficiently large values of x we obtain the asymptotic representation

$$H_\nu^{(1)}(x) = \sqrt{\frac{2}{\pi x}}\, e^{i[x-(\pi/2)(\nu+1/2)]} + \cdots \qquad \text{(A-2.152)}$$

Analogously, for $H_\nu^{(2)}(x)$ we obtain the asymptotic representation

$$H_\nu^{(2)}(x) = \sqrt{\frac{2}{\pi x}}\, e^{-i[x-(\pi/2)(\nu+1/2)]} + \cdots. \qquad \text{(A-2.153)}$$

[55] The error obtained by replacing the finite limits of the integral with infinite limits decreases exponentially since for large values of z

$$\int_z^\infty e^{\xi^2}\, d\xi \approx \frac{e^{-z^2}}{2z}.$$

We can be certain of this if we consider the quotient

$$\int_z^\infty e^{-\xi^2}\, d\xi : \frac{e^{-z^2}}{2z}$$

and determine its limit value.

Equations (A-2.152) and (A-2.153) have an approximate character and the omitted terms are of order $x^{-3/2}$. The next term of the expansion can be obtained if in the expressions for $-i\sin\varphi$ and $e^{i\nu\varphi}$ we also use terms which, with respect to s, are small and of higher order.

From the formulas for $H_\nu^{(1)}(x)$ and $H_\nu^{(2)}(x)$, the asymptotic representations

$$J_\nu(x) = \frac{1}{2}\{H_\nu^{(1)}(x) + H_\nu^{(2)}(x)\} \approx \sqrt{\frac{2}{\pi x}}\cos\left[x - \frac{\pi}{2}\left(\nu + \frac{1}{2}\right)\right] + \cdots, \quad \text{(A-2.154)}$$

$$N_\nu(x) = \frac{1}{2i}\{H_\nu^{(1)}(x) - H_\nu^{(2)}(x)\} \approx \sqrt{\frac{2}{\pi x}}\sin\left[x - \frac{\pi}{2}\left(\nu + \frac{1}{2}\right)\right] + \cdots \quad \text{(A-2.155)}$$

are obtained for J_ν and N_ν. Furthermore, we indicated earlier that we proved formula (A-2.154) in Section A-2.3(2) for integers $\nu = n$.

In support of the applicability of the asymptotic representations for all values of ν, we have convinced ourselves that the cylindrical functions introduced by means of contour integrals and cylindrical functions introduced by means of series expansions are identical. We shall not concern ourselves with the proof of this assertion.

Finally, we should note that the above described saddle point method is suitable for obtaining asymptotic representations for a series of other functions which are represented by contour integrals. In particular, this is possible for $H_\nu^{(1,2)}(x)$ for $\nu \approx x \to \infty$.

A-3. SPHERICAL FUNCTIONS

Spherical functions are introduced in connection with the investigation of the solution of the Laplace differential equation and, in particular, with potential theory. In Section A-3.1 we consider the Legendre polynomials, which are then used in Section A-3.2 for the construction of spherical functions. Spherical functions represent an important instrument for the solution of many problems of mathematical physics.

1. Legendre Polynomials

1. *The generating functions and the Legendre polynomials.* The Legendre polynomials are closely related to the fundamental solution $1/R$ of the Laplace differential equation. Here R is the distance of the test point M from the source point M_0. Let r and r_0 be the radius vectors of the points M and M_0 and θ the angle enclosed by them (Figure 96). Obviously, then, we can write

$$\frac{1}{R} = \frac{1}{\sqrt{r_0^2 + r_2 - 2rr_0\cos\theta}} = \begin{cases} \dfrac{1}{r_0}\dfrac{1}{\sqrt{1+\rho^2-2\rho x}} & \text{for } r < r_0, \\ \dfrac{1}{r}\dfrac{1}{\sqrt{1+\rho^2-2\rho x}} & \text{for } r > r_0, \end{cases} \quad \text{(A-3.1)}$$

where $x = \cos\theta$ $(-1 \leqq x \leqq 1)$, and $\rho = r/r_0 < 1$ or $\rho = r_0/r < 1$ is valid (in both cases ρ is smaller than 1).

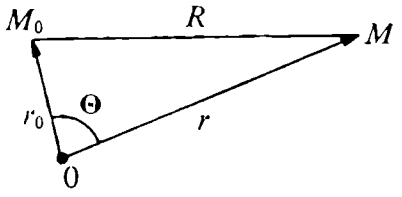


FIG. 96

The function

$$\Psi(\rho, x) = \frac{1}{\sqrt{1 + \rho^2 - 2\rho x}} \qquad (0 < \rho < 1, \ -1 \leqq x \leqq 1) \tag{A-3.2}$$

is denoted as the generating function of the Legendre polynomials.

We now expand the function $\Psi(\rho, x)$ in a series in powers of ρ:

$$\Psi(\rho, x) = \sum_{n=0}^{\infty} P_n(x)\rho^n \ . \tag{A-3.3}$$

Then the coefficients $P_n(x)$ in the expansion (A-3.3), as we shall see, represent polynomials of the nth degree and are called Legendre polynomials.

This expansion corresponds to the expansion

$$\frac{1}{R} = \begin{cases} \frac{1}{r_0} \sum_{n=0}^{\infty} \left(\frac{r}{r_0}\right)^n P_n(\cos\theta) & \text{for } r < r_0 \ , \\ \frac{1}{r} \sum_{n=0}^{\infty} \left(\frac{r_0}{r}\right)^n P_n(\cos\theta) & \text{for } r > r_0 \ . \end{cases} \tag{A-3.4}$$

By considering the expansion

$$\begin{aligned} (1 + \rho^2 - 2\rho x)^{-1/2} &= 1 - \frac{1}{2}(\rho^2 - 2\rho x) + \frac{3}{8}(\rho^2 - 2\rho x)^2 + \cdots \\ &= 1 + \rho x + \rho^2\left(\frac{3}{2}x^2 - \frac{1}{2}\right) + \cdots + \rho^n(\alpha_0 x^n + \cdots = \sum_{n=0}^{\infty} P_n(x)\rho^n \end{aligned} \tag{A-3.5}$$

for sufficiently small values of ρ, we can be certain that each of the functions $P_n(x)$ really is a polynomial of the nth degree. For $k \leqq n$, therefore, we obtain all terms which contain the power ρ^n in the brackets of the form $(\rho^2 - 2\rho x)^k$, where x^n occurs only in the term for $k = n$. If $k < n$, then the coefficient of ρ^n contains only powers of x of smaller degree than n. Since the general term of the polynomials $(\rho^2 - 2\rho x)^k$ has the form $A_m\rho^{2m}(\rho x)^{k-m} = A_m\rho^{m+k}x^{k-m}$, where A_m denotes fixed coefficients, even powers of x occur only for even powers of ρ and odd powers of x only for odd powers of ρ. Therefore, each polynomial $P_n(x)$ contains either only even or only odd powers of x, according to whether n is even or odd.

In particular, it follows that

$$P_n(-x) = (-1)^n P_n(x) \ . \tag{A-3.6}$$

For $x = 1$ the generating function $\Psi(\rho, x)$ equals

$$\frac{1}{1 - \rho} = 1 + \rho + \rho^2 + \cdots + \rho^n + \cdots = \sum_{n=0}^{\infty} P_n(1)\rho^n \ , \tag{A-3.7}$$

that is,

$$P_n(1) = 1 \tag{A-3.8}$$

for all values of n. From Eqs. (A-3.6) and (A-3.8) there results

$$P_n(-1) = (-1)^n . \tag{A-3.9}$$

2. *A recursion formula.* By differentiation of $\Psi(\rho, x)$ with respect to ρ, we easily obtain the expression

$$(1 - 2\rho x + \rho^2)\frac{\partial \Psi}{\partial \rho} - (x - \rho)\Psi = 0 . \tag{A-3.10}$$

We shall write the left side of this formula as a power series in powers of ρ. To do this, we substitute the series (A-3.3) for Ψ and the series

$$\frac{\partial \Psi}{\partial \rho} = P_1(x) + 2P_2(x)\rho + 3P_3(x)\rho^2 + \cdots$$

into expression (A-3.10).

The coefficient of ρ^n in the series thus obtained must be equal to zero for all x because of (A-3.10):

$$(n + 1)P_{n+1}(x) - x(2n + 1)P_n(x) + nP_{n-1}(x) = 0 . \tag{A-3.11}$$

This is the recursion formula sought, which links the three successive Legendre polynomials with each other. It can be used for the calculation of $P_n(x)$ $(n \geqq 2)$, if we consider that $P_0(x) = 1$ and $P_1(x) = x$.

We shall now calculate the coefficients a_n of the highest power x^n in the polynomial $P_n(x)$. From formula (A-3.11) we obtain

$$a_{n+1} = \frac{2n + 1}{n + 1} a_n . \tag{A-3.12}$$

Here, if we set $a_0 = 1$, we obtain, successively, $a_1 = 1$, $a_2 = \frac{3}{2}$, $a_3 = \frac{5}{3}$, and so forth. By complete induction we prove the general formula

$$a_n = \frac{1 \cdot 3 \cdot 5 \cdots (2n - 1)}{n!} . \tag{A-3.13}$$

3. *The Legendre differential equation.* In this section we shall show that $P_n(x)$ in the region $-1 < x < 1$ for $\lambda = n(n + 1)$ represents a bounded solution of the Legendre differential equation

$$\frac{d}{dx}\left[(1 - x^2)\frac{dy}{dx}\right] + \lambda y = 0$$

or

$$(1 - x^2)y'' - 2xy' + \lambda y = 0 .$$

By differentiation of

$$\Psi(\rho, x) = \sum_{n=0}^{\infty} P_n(x)\rho^n$$

with respect to x, we obtain the equation

$$(1 - 2x\rho + \rho^2)\frac{\partial \Psi}{\partial x} - \rho\Psi = 0 . \tag{A-3.14}$$

Together with Eq. (A-3.10), we obtain

$$\rho\frac{\partial\Psi}{\partial\rho} - (x-\rho)\frac{\partial\Psi}{\partial x} = 0\,. \tag{A-3.15}$$

As in the above expression, we obtain from this the second recursion formula

$$nP_n(x) - xP_n'(x) + P_{n-1}'(x) = 0\,. \tag{A-3.16}$$

By substitution of the expression for $\rho\Psi$ and $\rho(\partial\Psi/\partial\rho)$, obtained from Eqs. (A-3.14) and (A-3.15), into the identity

$$\rho\frac{\partial}{\partial\rho}(\partial\Psi) = \rho\left(\rho\frac{\partial\Psi}{\partial\rho} + \Psi\right) \tag{A-3.17}$$

we obtain

$$\rho\frac{\partial}{\partial\rho}(\rho\Psi) - (1-\rho x)\frac{\partial\Psi}{\partial x} = 0\,. \tag{A-3.18}$$

Again, we expand the left side of Eq. (A-3.18) in a power series in powers of ρ. From the requirement that the coefficients of the powers ρ^n be equal to zero, the third recursion formula

$$nP_{n-1}(x) - P_n'(x) + xP_{n-1}'(x) = 0 \tag{A-3.19}$$

then follows.

We now eliminate $P_{n-1}(x)$ and $P_{n-1}'(x)$ from Eqs. (A-3.16) and (A-3.19). For this, we introduce in Eq. (A-3.19) the expression for $P_{n-1}'(x)$ from Eq. (A-3.16), then differentiate the expression obtained with respect to x and use once more formula (A-3.16). In this manner, we arrive at the equation

$$(1-x^2)P_n''(x) - 2xP_n'(x) + n(n+1)P_n(x) = 0\,. \tag{A-3.20}$$

Thus the Legendre polynomials are eigenfunctions of the following boundary-value problem:

Find those λ-values, for which, in the region $-1 < x < 1$, nontrivial solutions of the Legendre differential equation

$$\frac{d}{dx}\left[(1-x^2)\frac{dy}{dx}\right] + \lambda y = 0 \tag{A-3.21}$$

exist, that are bounded for $x = 1$ and satisfy the normalization condition $P_n(1) = 1$. The eigenvalue corresponding to $P_n(x)$ is $\lambda_n = n\,(n+1)$.

4. *The orthogonality of the Legendre polynomials.* The Legendre equation (A-3.21) appears as a special case of the equation

$$\frac{d}{dx}\left[k(x)\frac{dy}{dx}\right] - q(x)y + \lambda\rho y = 0 \tag{A-3.22}$$

with for $q = 0$, $\rho = 1$, $k(x) = 1 - x^2$. Therefore, the general theory developed for Eq. (A-3.22) applies to the Legendre equation. From this theory it follows that:

(a) The Legendre polynomials of different orders are orthogonal to each other in the interval $(-1, 1)$:

$$\int_{-1}^{1} P_n(x)P_m(x)\,dx = 0 \qquad \text{for } m \neq n\,. \tag{A 3.23}$$

(b) Each additional linearly independent solution of the Legendre equation for $\lambda = n(n+1)$ becomes infinite at the point $x = \pm 1$ as $\log|1 \mp x|$.

A system of orthogonal polynomials, as is well known, is closed.[56]

It now follows that the Legendre equation for $\lambda \neq n\,(n+1)$ can possess no nontrivial bounded solution $y(x)$. For if such a solution $y_n(x)$ were to exist for $\lambda \neq n\,(n+1)$, then it must be orthogonal to all $P_n(x)$. However, on the basis of the closedness of the system of orthogonal polynomials $P_m(x)$, it would follow that

$$y_n(x) \equiv 0\,.$$

[56] A system $\{\varphi_n\}$ of orthogonal functions is said to be closed if there exists no continuous function which does not vanish identically and is orthogonal to all functions of the system, that is, if the only continuous function which is orthogonal to every φ_n is the zero function.

A system of $\{\varphi_n\}$ of orthogonal functions is called complete in the interval $[a, b]$ if an arbitrary continuous function can be approximated by a linear combination of the functions φ_n in the mean with arbitrary small error. In other words, to each number $\varepsilon > 0$, a linear combination

$$S_n = c_1\varphi_1 + \cdots + c_n\varphi_n$$

can be found such that

$$\int_a^b [f(x) - S_n(x)]^2\,dx < \varepsilon\,.$$

For a complete system of functions $\{v_n\}$ the so-called completeness relation or Parseval equation holds:

$$\int_a^b f^2(x)\,dx = \sum_{n=1}^{\infty} N_n f_n^2\,,$$

where

$$f_n = \frac{1}{N_n}\int_a^b f(\xi)\varphi_n(\xi)\,d\xi$$

are the Fourier coefficients of the function $f(x)$.

Closedness is a consequence of completeness. Let $\varphi_n(x)$ be a complete system of orthogonal functions.

We assume that there exists a continuous function $f(x) \not\equiv 0$, which is orthogonal to all $\varphi_n(x)$. Then, because of the completeness of the system $\{\varphi_n\}$, the equation

$$\int_a^b f^2(x)\,dx = \sum_{n=1}^{\infty} N_n f_n^2 = 0$$

must hold since, by assumption, $f_n = 0$ for all n. However, from this, $f(x) \equiv 0$ follows, which contradicts our assumption, that is, the system $\{\varphi_n(x)\}$ is closed.

The completeness and, therefore, also the closedness of a system of orthogonal polynomials $P_n(x)$ are a consequence of the Weierstrass approximation theorem:

For each function $f(x)$ continuous in a closed interval $[a, b]$, there exists a polynomial $Q_n(x)$ which approximates $f(x)$ in $[a, b]$, arbitrarily closely, that is, for arbitrary $\varepsilon > 0$ and for all x in $[a, b]$,

$$|f(x) - Q_n(x)| < \varepsilon\,.$$

For proof of the completeness of the system of $P_n(x)$, we only need to represent the polynomial $Q_n(x)$ as a linear combination of the orthogonal polynomials $P_n(x)$ and apply the above inequality. Then the completeness relation for the system of orthogonal polynomials results directly.

5. *The norm of the Legendre polynomials.* We shall calculate the norm

$$N_n = \int_{-1}^{1} P_n^2(x)\,dx \tag{A-3.24}$$

of the Legendre polynomials. For this we consider the polynomial

$$Q(x) = P_n(x) - \frac{a_n}{a_{n-1}} xP_{n-1}(x)\,, \tag{A-3.25}$$

where a_n denotes the coefficient of x^n in $P_n(x)$ and a_{n-1} the coefficient of x^{n-1} in $P_{n-1}(x)$. $Q(x)$ is a polynomial of degree $n-2$ and can be written in the form

$$Q(x) = \sum_{k=0}^{n-2} A_k P_k(x)\,.$$

The coefficients can be determined successively beginning with $k = n-2$. Here we compare the coefficients of the highest powers of the polynomial $P_k(x)$.

It then follows that $Q(x)$ is orthogonal to $P_n(x)$:

$$\int_{-1}^{1} P_n(x)Q(x)\,dx = 0\,. \tag{A-3.26}$$

Therefore,

$$N_n = \int_{-1}^{1} P_n(x)\left[\frac{a_n}{a_{n-1}} xP_{n-1}(x) + Q(x)\right] dx = \frac{a_n}{a_{n-1}} \int_{-1}^{1} xP_n(x)P_{n-1}(x)\,dx\,.$$

Now by using the first recursion formula

$$xP_n(x) = \frac{n+1}{2n+1} P_{n+1}(x) + \frac{n}{2n+1} P_{n-1}(x)\,,$$

we find

$$N_n = \frac{n}{2n+1} \frac{a_n}{a_{n-1}} N_{n-1}\,. \tag{A-3.27}$$

Since, because of Eq. (A-3.12),

$$\frac{a_n}{a_{n-1}} = \frac{2n-1}{n}\,,$$

we find

$$N_n = \frac{2n-1}{2n+1} N_{n-1}\,. \tag{A-3.28}$$

By consideration of

$$P_0(x) = 1\,, \qquad N_0 = 2\,,$$

we find successively from Eq. (A-3.28) that

$$N_1 = \tfrac{2}{3}\,, \qquad N_2 = \tfrac{2}{5}$$

and, in general, by complete induction that

$$N_n = \frac{2}{2n+1}\,. \tag{A-3.29}$$

Consequently, the Legendre polynomial $P_n(x)$ forms an orthogonal system with norm $N_n = 2/(2n+1)$:

$$\int_{-1}^{1} P_m(x)P_n(x)\,dx = \begin{cases} 0 & \text{for } m \neq n\,, \\ \dfrac{2}{2n+1} & \text{for } m = n\,. \end{cases} \tag{A-3.30}$$

6. *A differential relation for the Legendre polynomials.* Now we shall prove that the Legendre polynomials are representable in the form

$$P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}[(x^2-1)^n]\,. \tag{A-3.31}$$

This formula is usually called the formula of Rodrigues.

For the proof of formula (A-3.31), it is sufficient to prove the following facts:

(a) $\bar{P}_n(x) = 1/2^n n!\; d^n/dx^n[(x^2-1)^n]$ is a solution of the Legendre equation.

(b) $\bar{P}_n(1) = 1$.

If we set

$$u = (x^2-1)^n\,,$$

then

$$u' = 2nx(x^2-1)^{n-1}\,.$$

Therefore, we also find

$$(x^2-1)u' - 2nxu = 0\,.$$

By $(m+1)$ times differentiation of this equation, we obtain

$$(x^2-1)u^{(m+2)} - (2n-2m-2)xu^{(m+1)} + [m(m+1) - 2n(m+1)]u^{(m)} = 0\,.$$

If we set $m = n$ and consider that the expression in the square brackets equals $-n(n+1)$, then we obtain

$$(1-x^2)u^{(n+2)} - 2xu^{n+1} + n(n+1)u^{(n)} = 0\,,$$

that is, the function

$$\bar{P}_n = \frac{1}{2^n n!}\frac{d^n u}{dx^n}$$

satisfies the Legendre equation

$$(1-x^2)\bar{P}_n'' - 2x\bar{P}_n' + n(n+1)\bar{P}_n = 0\,.$$

Consequently,

$$\bar{P}_n(x) = C_n P_n(x)\,, \tag{A-3.31'}$$

where C_n denotes a fixed constant.

Now we shall show that $\bar{P}_n(1) = 1$. For this we consider the derivative

$$\frac{d^m}{dx^m}[(x^2-1)^n] = \frac{d^m}{dx^m}[(x+1)^n(x-1)^n] = c_0(x+1)^{n-m}(x-1)^n + c_1(x+1)^{n-m+1}(x-1)^{n-1} + \cdots + c_n(x+1)^n(x-1)^{n-m}\,.$$

If $m < n$, then all summands vanish at the points $x = \pm 1$:

$$\left\{\frac{d^n}{dx^n}[(x^2-1)^n]\right\}_{x=\pm 1} = 0\,, \qquad (m < n)\,.$$

However, if $m = n$, then

$$\left\{\frac{d^m}{dx^m}[(x^2-1)^n]\right\}_{x=1} = 2^n n!$$

from which it follows that

$$\bar{P}_n(1) = 1\,.$$

Now if in Eq. (A-3.31′) we set $x = 1$, then we obtain

$$C_n = 1\,,$$

that is,

$$\bar{P}_n(x) \equiv P_n(x)\,.$$

Thus we have

$$P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}[(x^2-1)^n]\,.$$

With the help of this formula, we can easily prove the following important properties of the Legendre polynomials:

The Legendre polynomial $P_n(x)$ possesses exactly n zeros within the interval $(-1, +1)$. Let the kth derivative $(d^k/dx^k)P_n(x)$ $(k \leqq n)$ have exactly $n - k$ zeros within $(-1, +1)$ and at the end points let it be different from zero.

The function $(t^2-1)^n$ vanishes at the end points of the interval $[-1, 1]$. Its first derivative vanishes at the ends of the interval. Based on Rolle's theorem, the derivative must possess at least one zero within the interval. The second derivative has at least two zeros within the interval and vanishes, moreover, at the ends of the interval (Figure 97). By repetition of these

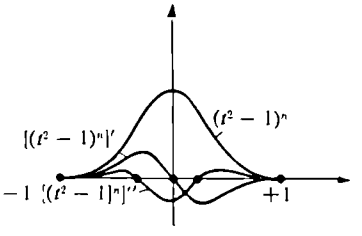


FIG. 97

results, we find that the nth derivative has at least n zeros in the interval $(-1, 1)$. There, indeed, it has exactly n zeros since it can be treated as a polynomial of the nth degree. Thus, the first part of the above proposition is proved. The derivative $(d/dx)\,P_n(x)$ must, on the basis of these theorems, possess at least $n - 1$ zeros within the interval. However, since the derivative represents a polynomial of degree $n - 1$, it must possess exactly $n - 1$ zeros within the interval. Correspondingly, $(d^k/dx^k)P_n$ has exactly $n - k$ zeros within the interval. Therefore, the above statements are completely proved.

7. *An integral formula. The boundedness of the Legendre polynomials.* The Legendre polynomials $P_n(\mu)$ are bounded for all values of the argument $-1 \leqq \mu \leqq +1$. This shall now be proved.

First, we consider the integral

$$I = \int_0^{\pi} \frac{d\varphi}{a - ib\cos\varphi}$$

in which a and b denote any constants. To calculate these integrals, we make the following transformation:

$$\begin{aligned} I &= a\int_0^{\pi} \frac{d\varphi}{a^2 + b^2\cos^2\varphi} + ib\int_0^{\pi} \frac{\cos\varphi\, d\varphi}{a^2 + b^2\cos^2\varphi} \\ &= a\int_0^{\pi} \frac{d\varphi}{a^2 + b^2\cos^2\varphi} - ib\int_{-\pi/2}^{\pi/2} \frac{\sin\psi\, d\psi}{a^2 + b^2\sin^2\psi} = I_1 - I_2 , \qquad \text{(A-3.32)} \\ &\left(\psi = \varphi - \frac{\pi}{2}\right). \end{aligned}$$

The integral I_2 is equal to zero since the integrand is an odd function. The calculation of the integral I_1 gives

$$\begin{aligned} I_1 &= \frac{\pi}{\sqrt{a^2 + b^2}} \qquad \text{for } a > 0 , \\ &= -\frac{\pi}{\sqrt{a^2 + b^2}} \qquad \text{for } a < 0 . \end{aligned} \qquad \text{(A-3.32')}$$

Now if we set

$$a = 1 - x\mu,\ b = x\sqrt{1 - \mu^2} , \qquad 0 < x < 1,\ -1 \leqq \mu \leqq 1)$$

and use formula (A-3.32'), then we obtain

$$\frac{1}{\pi}\int_0^{\pi} \frac{d\varphi}{1 - x(\mu + i\sqrt{1 - \mu^2}\cos\varphi)} = \frac{1}{\sqrt{1 + x^2 - 2x\mu}} . \qquad \text{(A-3.33)}$$

The generating function standing on the right side of this equation serves to define the Legendre polynomials as follows:

$$\frac{1}{\sqrt{1 + x^2 - 2x\mu}} = \sum_{n=0}^{\infty} x^n P_n(\mu) .$$

Now we expand the integrand in the series

$$\frac{1}{1 - x(\mu + i\sqrt{1 - \mu^2}\cos\varphi)} = \sum_{n=0}^{\infty} x^n(\mu + i\sqrt{1 - \mu^2}\cos\varphi)^n . \qquad \text{(A-3.34)}$$

Since

$$|\, x(\mu + i\sqrt{1 - \mu^2}\cos\varphi)\,| = x\sqrt{\mu^2\sin^2\varphi + \cos^2\varphi} \leqq x < 1 ,$$

series (A-3.34) converges uniformly. It can be integrated termwise:

$$\frac{1}{\pi}\int_0^{\pi} \frac{1}{1 - x(\mu + i\sqrt{1 - \mu^2}\cos\varphi)}\, d\varphi = \sum_{n=0}^{\infty} x^n \cdot \frac{1}{\pi}\int_0^{\pi} (\mu + i\sqrt{1 - \mu^2}\cos\varphi)^n\, d\varphi .$$

On the other hand, because of Eqs. (A-3.3) and (A-3.17), we obtain

$$\frac{1}{\sqrt{1 + x^2 - 2x\mu}} = \sum_{n=0}^{\infty} x^n P_n(\mu) = \sum_{n=0}^{\infty} x^n \cdot \frac{1}{\pi}\int_0^{\pi} (\mu + i\sqrt{1 - \mu^2}\cos\varphi)^n\, d\varphi .$$

From this we obtain the integral representation

$$P_n(\mu) = \frac{1}{\pi}\int_0^{\pi} (\mu + i\sqrt{1-\mu^2}\cos\varphi)^n \, d\varphi , \qquad (-1 \leqq \mu \leqq 1) \qquad \text{(A-3.35)}$$

for the Legendre polynomials. If $-1 < \mu < 1$, then

$$|\mu + i\sqrt{1-\mu^2}\cos\varphi| = \sqrt{\mu^2\sin^2\varphi + \cos^2\varphi} \leqq 1 ,$$

and further, for $P_n(\mu)$ the inequality

$$|P_n(\mu)| \leqq \frac{1}{\pi}\int_0^{\pi} d\varphi = 1 \qquad \text{for } -1 \leqq \mu \leqq 1$$

holds. Here the equality signs are valid only for $\mu = \pm 1$.

The Legendre polynomials, therefore, are bounded, and in fact,

$$|P_n(\mu)| < 1 \qquad \text{(A 3.36)}$$

holds for all μ between -1 and $+1$.

8. *The associated Legendre polynomials.* Together with the Legendre polynomials $P_n(x)$ the so-called associated Legendre polynomials $P_n^{(m)}(x)$ play a large role. They are the solutions of the equation

$$\frac{d}{dx}\left[(1-x^2)\cdot\frac{dz}{dx}\right] + \left(\lambda - \frac{m^2}{1-x^2}\right)z = 0 \qquad \text{(A 3.37)}$$

or

$$(1-x^2)z'' - 2xz' + \left(\lambda - \frac{m^2}{1-x^2}\right)z = 0 , \qquad \text{(A 3.37')}$$

which are bounded in the interval $(-1, 1)$.

The associated polynomial of mth order can be defined through the Legendre polynomial as

$$P_n^{(m)}(x) = (1-x^2)^{m/2}\frac{d^m}{dx^m}P_n(x) . \qquad \text{(A 3.38)}$$

We shall now prove the validity of this relation.

The equation for the associated Legendre polynomials (A 3.37') with the substitution

$$z = (1-x^2)^{m/2}Y(x)$$

assumes the form

$$(1-x^2)Y'' - 2(m+1)xY' + [\lambda - m(m+1)]Y = 0 . \qquad \text{(A 3.39)}$$

We shall now show that Eq. (A 3.39) is satisfied by the mth derivative of the corresponding Legendre polynomials. We differentiate the Legendre differential equation (A 3.21) m times with respect to x. After some simplification, we obtain

$$(1-x^2)\frac{d^{m+2}y}{dx^{m+2}} - 2(m+1)x\frac{d^{m+1}y}{dx^{m+1}} + [\lambda - m(m+1)]\frac{d^m y}{dx^m} = 0 . \qquad \text{(A 3.40)}$$

By comparison of these results with formula (A-3.39), we recognize that the function

$$Y(x) = \frac{d^m y(x)}{dx^m}$$

is a particular solution of Eq. (A-3.39). Consequently, the expression

$$Z = (1 - x^2)^{m/2} Y(x) = (1 - x^2)^{m/2} \frac{d^m}{dx^m} y(x) \tag{A-3.41}$$

satisfies the equation for the associated polynomials and is bounded for $x = \pm 1$.

For

$$\lambda = n(n + 1) \qquad \text{and} \qquad y = P_n(x)$$

formula (A-3.41) represents the associated polynomial of mth order. From this formula, it is obvious that $P_n^{(m)}(x)$ is different from zero only for $m \leqq n$.

The associated Legendre polynomials form, on the basis of the general theory, an orthogonal system. Now we shall calculate the norm of the associated polynomials. At the same time we shall prove its orthogonality. We multiply Eq. (A-3.40) by $(1 - x^2)^m$ and write the result in the form

$$\frac{d}{dx}\left[(1 - x^2)^{m+1} \frac{d^{m+1}}{dx^{m+1}} P_n\right] = -[\lambda - m(m + 1)](1 - x^2)^m \frac{d^m P_n}{dx^m} .$$

If we set $m' = m + 1$, then we obtain

$$\frac{d}{dx}\left[(1 - x^2)^{m'} \frac{d^{m'} P_n}{dx^{m'}}\right] = -[\lambda - m'(m' - 1)](1 - x^2)^{m'-1} \frac{d^{m'-1} P_n}{dx^{m'-1}} . \tag{A-3.42}$$

We now introduce the relation

$$L^m_{n,k} = \int_{-1}^{+1} P_n^{(m)}(x) P_k^{(m)}(x)\, dx = \int_{-1}^{+1} (1 - x^2)^m \frac{d^m P_n}{dx^m} \frac{d^m P_k}{dx^m}\, dx .$$

Partial integration of the right side gives

$$L^m_{n,k} = \left[\frac{d^{m-1} P_k}{dx^{m-1}} \frac{d^m P_n}{dx^m} (1 - x^2)^m\right]_{-1}^{+1} - \int_{-1}^{+1} \frac{d^{m-1} P_k}{dx^{m-1}} \frac{d}{dx}\left[(1 - x^2)^m \frac{d^m P_n}{dx^m}\right] dx .$$

The first term on the right side is equal to zero. If we transform the integral remaining on the right by taking into consideration the differential expression (A-3.42), then we obtain

$$\begin{aligned} L^m_{n,k} &= [n(n + 1) - m(m - 1)] \int_{-1}^{+1} (1 - x^2)^{m-1} \frac{d^{m-1}}{dx^{m-1}} P_n \frac{d^{m-1}}{dx^{m-1}} P_k\, dx \\ &= (n + m)(n - m + 1) L^{m-1}_{n,k} . \end{aligned}$$

By repetition of the above, we finally arrive at the formula

$$\begin{aligned} L^m_{n,k} &= (n + m)(n + m - 1) \cdots (n + 1) n \cdots (n - m + 1) L^0_{n,k} \\ &= \frac{(n + m)!}{n!} \frac{n!}{(n - m)!} L^0_{n,k} = \frac{(n + m)!}{(n - m)!} L^0_{n,k} . \end{aligned}$$

Hence, by using the equation

$$L^0_{n,k} = \int_{-1}^{1} P_n(x)P_k(x)\,dx = \begin{cases} 0 & \text{for } k \neq n\,, \\ \dfrac{2}{2n+1} & \text{for } k = n \end{cases}$$

[see system (A-3.30)], we obtain the integral relation

$$\int_{-1}^{1} P_n^{(m)}(x)P_k^{(m)}(x)\,dx = \begin{cases} 0 & \text{for } k \neq n\,, \\ \dfrac{2}{2n+1}\dfrac{(n+m)!}{(n-m)!} & \text{for } k = n\,, \end{cases} \tag{A-3.43}$$

that is, the associated polynomials are orthogonal to each other and have a norm given by

$$\int_{-1}^{1} [P_n^{(m)}(x)]^2\,dx = \frac{2}{2n+1}\frac{(n+m)!}{(n-m)!}\,. \tag{A-3.43'}$$

9. *The closedness of the systems of associated Legendre polynomials.* We shall prove that through the system of associated polynomials $P_n^{(m)}(x)$ all bounded solutions of Eq. (A-3.37) are included.

For $\lambda = n(n+1)$, each solution which is linearly independent of $P_n^{(m)}(x)$ is infinite at the point $x = +1$. For $\lambda \neq n(n+1)$, however, each bounded solution must be orthogonal to all $P_n^{(m)}(x)$.

We now show that no bounded solution of Eq. (A-3.37) different from $P_n^{(m)}(x)$ exists. For this it is sufficient to show that the system $\{P_n^{(m)}(x)\}$ of associated polynomials is closed, that is, there exists no non-identically vanishing continuous function which is orthogonal to all functions of the system.

Lemma. Each function $f(x)$ continuous in the interval $[-1, 1]$, which is equal to zero at the end points $x = 1$ and $x = -1$, can be approximated with arbitrary accuracy uniformly by the associated Legendre polynomials of arbitrary order m.

Proof. We note first that the derivatives of the Legendre polynomial $(d^m/dx^m)P_n(x)$ are polynomials of degree $n - m$. Since an arbitrary polynomial in x can now be represented by a linear combination of these polynomials, and because of the Weierstrass approximation theorem, an arbitrary function $f(x)$ which is continuous in the interval $[-1, 1]$ can be approximated uniformly by linear combinations of the derivatives $(d^m/dx^m)P_n(x)$:

$$\left| \tilde{f}(x) - \sum_{n}^{n_0} \sum_{m} c_n \frac{d^m}{dx^m} P_n(x) \right| < \varepsilon\,, \quad \text{if } n_0 > N(\varepsilon)\,. \tag{A-3.44}$$

By multiplying this inequality by $(1 - x^2)^{m/2}$, we obtain

$$\left| f_1(x) - \sum_{n=m}^{n_0} c_n P_n^{(m)}(x) \right| < \varepsilon\,, \quad \text{if } n_0 > N(\varepsilon)\,, \tag{A-3.45}$$

where

$$f_1(x) = \tilde{f}(x)(1 - x^2)^{m/2}\,; \tag{A-3.46}$$

that is, an arbitrary function $f_1(x)$, which is represented in the form (A-3.46) by a function $f(x)$ continuous in $[-1, 1]$, can be approximated with arbitrary accuracy uniformly by linear combinations of associated Legendre polynomials.

We say that a function $f_1(x)$ belongs to class H_1 if it is continuous in the interval $(-1, 1)$ and vanishes identically in a small neighborhood of the point $x = \pm 1$:

$$f_1(x) = 0 \quad \text{for } |1 - \delta| \leqq |x| \leqq 1 .$$

Since for each function $f_1(x)$ belonging to the class H_1 the function

$$\bar{f}(x) = \frac{f_1(x)}{(1 - x^2)^{m/2}}$$

is continuous in $[-1, 1]$, our lemma is proved for a function of class H_1.

Now we consider an arbitrary function $f(x)$ which is continuous in $[-1, 1]$ and vanishes at the ends of the interval. Obviously, such a function can be approximated uniformly by a function $f_1(x)$ from the class H_1 with an accuracy of $\varepsilon/2$:

$$|f(x) - f_1(x)| < \frac{\varepsilon}{2} .$$

If we now approximate $f_1(x)$ by a linear combination of the associated polynomials with an accuracy of $\varepsilon/2$,

$$|f_1(x) - \Sigma_1(x)| < \frac{\varepsilon}{2} , \qquad \Sigma_1(x) = \sum_{n}^{n_0} \sum_{m} c_n P_n^{(m)}(x) ,$$

then we obtain the inequality

$$|f(x) - \Sigma_1(x)| < \varepsilon$$

with which our lemma is proved.

We can now easily prove the completeness of the system of associated polynomials and, thereby, also the closedness.

We recall that a system of functions $\{\varphi_n(x)\}$ is called complete in the interval $[a, b]$ if an arbitrary function $F(x)$, which is continuous in $[a, b]$, can be approximated in the mean with arbitrary accuracy by linear combinations of these functions:

$$\int_a^b [F(x) - \sum_{n=1}^{n_0} c_n \varphi_n(x)]^2 \, dx < \varepsilon \quad \text{for } n_0 > N(\varepsilon) .$$

Obviously, each function $F(x)$ continuous in $[-1, 1]$ can be approximated in the mean with arbitrary accuracy by a function $f(x)$ which is continuous in $[-1, 1]$ and which, for $x = \pm 1$, is equal to zero:

$$\int_{-1}^{1} [F(x) - f(x)]^2 \, dx < \varepsilon' .$$

Now we take a linear combination of associated polynomials which approximates $f(x)$ uniformly:

$$|f(x) - \Sigma_1(x)| < \varepsilon'' .$$

Then, using the inequality

$$(a + b)^2 \leqq 2(a^2 + b^2) ,$$

we obtain

$$\int_{-1}^{1} [F(x) - \Sigma_1]^2 \, dx \leqq 2\int_{-1}^{1} [F(x) - f(x)]^2 \, dx + 2\int_{-1}^{1} [f(x) - \Sigma_1]^2 \, dx < \varepsilon ,$$

$$\text{for } 2\varepsilon' + 4(\varepsilon'')^2 \leqq \varepsilon ,$$

whereby the completeness and, at the same time, the closedness of the system of associated polynomials is proved.

2. Harmonic polynomials and spherical functions

1. *Harmonic polynomials.* A homogeneous polynomial which satisfies the Laplace equation

$$\Delta u = u_{xx} + u_{yy} + u_{zz} = 0 \tag{A-3.47}$$

is called a harmonic polynomial.

As we can easily prove, the first two homogeneous harmonic polynomials have the form

$$u_1(x, y, z) = Ax + By + Cz ,$$
$$u_2(x, y, z) = Ax^2 + By^2 - (A + B)z^2 + Cxy + Dxz + Eyz ,$$

where A, B, C, D, E denote arbitrary coefficients.

We shall now calculate the number of linearly independent, homogeneous, harmonic polynomials

$$u_n = \sum_{p+q+r=n} \alpha_{p,q,r} x^p y^q z^r \tag{A-3.48}$$

of degree n. A complete homogeneous function of the nth degree has $[(n + 1)(n + 2)]/2$ coefficients. Therefore, the right side of Eq. (A-3.48) can be written in the form

$$\begin{aligned} &\alpha_{0,0,n} z^n + (\alpha_{1,0,n-1} x + \alpha_{0,1,n-1} y) z^{n-1} \cdots \\ &\quad + (\alpha_{n-1,0,1} x^{n-1} + \alpha_{n-2,1,1} x^{n-2} y + \cdots + \alpha_{0,n-1,1} y^{n-1}) z \\ &\quad + (\alpha_{n,0,0} x^n + \alpha_{n-1,1,0} x^{n-1} y + \cdots + \alpha_{0,n,0} y^n) z^0 . \end{aligned}$$

For z^n one coefficient appears, for z^{n-1} there are two, $\cdots$, for z, finally, there are n coefficients while for z^0 there are altogether $(n + 1)$ coefficients. The total number of coefficients, therefore, is equal to

$$1 + 2 + \cdots + n + (n + 1) = \frac{(n + 1)(n + 2)}{2} . \tag{A-3.49}$$

Eq. (A-3.47) imposes on the coefficients $[n(n - 1)]/2$ linear homogeneous relations since ΔU_n represents a homogeneous function of degree $n - 2$. Therefore, the polynomial must possess not less than $[(n + 1)(n + 2)]/2 - [(n - 1)n]/2 = 2n + 1$ linearly independent coefficients. For the case in which the $[(n - 1)n]/2$ relations are linearly independent, the number of independent coefficients must be larger than $2n + 1$.

We shall now show that only $2n + 1$ coefficients are linearly independent. The coefficients $\alpha_{p,q,r}$ of a homogeneous polynomial can be written in the form

$$\alpha_{p,q,r} = \frac{1}{p!q!r!} \frac{\partial^n u_n}{\partial x^p \partial y^q \partial z^r} .$$

Now, if u_n is a harmonic polynomial, then $\alpha_{p,q,r}$ for $r \geqq 2$ can be expressed by the coefficients $\alpha_{p,q,0}$ and $\alpha_{p,q,1}$ whose number is exactly equal to $2n + 1$.

Therefore,

$$\begin{aligned}\alpha_{p,q,r} &= \frac{1}{p!q!r!} \frac{\partial^{n-2}}{\partial x^p \partial y^q \partial z^{r-2}} \left[\frac{\partial^2 u_n}{\partial z^2}\right] \\ &= \frac{1}{p!q!r!} \frac{\partial^{n-2}}{\partial x^p \partial y^q \partial z^{r-2}} \left[-\frac{\partial^2 u_n}{\partial x^2} - \frac{\partial^2 u_n}{\partial y^2}\right] \\ &= \beta_1 \alpha_{p+2,q,r-2} + \beta_2 \alpha_{p,q+2,r-2} .\end{aligned}$$

If we now proceed correspondingly with the coefficients $\alpha_{p+2,q,r-2}$ and $\alpha_{p,q+2,r-2}$, then we have expressed the last $\alpha_{p,q,r}$ in terms of coefficients of the type $\alpha_{p,q,0}$ $(p + q = n)$ and $\alpha_{p,q,1}$ $(p + q + 1 = n)$. The number of coefficients of the type $\alpha_{p,q,0}$ however, is equal to $n + 1$, and the number of coefficients of type $\alpha_{p,q,1}$ equals n. Thus, the total number of linearly independent coefficients and, consequently, the linearly independent harmonic polynomials of the nth degree exactly equal $2n + 1$.

We call all rational spherical functions (and also general spherical functions) homogeneous harmonic polynomials.

2. *Spherical functions.* Spherical functions can be introduced simply in the solution of the Laplace differential equation for a spherical region by means of separation of variables.

For this purpose we write the Laplace differential equation in polar coordinates

$$\Delta u = \frac{1}{r^2} \frac{\partial}{\partial r}\left(r^2 \frac{\partial u}{\partial r}\right) + \frac{1}{r^2 \sin\theta} \frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial u}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta} \frac{\partial^2 u}{\partial \varphi^2} = 0 , \quad \text{(A-3.47)}$$

and for the solution make the Ansatz

$$u(r, \theta, \varphi) = R(r)\, Y(\theta, \varphi) .$$

For the determination of $R(r)$ we obtain the Euler differential equation

$$r^2 R'' + 2rR' - \lambda R = 0 , \quad \text{(A-3.50)}$$

and for the determination of $Y(\theta, \varphi)$ we obtain the equation

$$\Delta_{\theta,\varphi} Y + \lambda Y = \frac{1}{\sin\theta} \frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial Y}{\partial \theta}\right) + \frac{1}{\sin^2\theta} \frac{\partial^2 Y}{\partial \varphi^2} + \lambda Y = 0 \quad \text{(A-3.51)}$$

with the auxiliary condition that $Y(\theta, \varphi)$ is bounded on the entire sphere.

In particular $Y(\theta, \varphi)$ satisfies the condition

$$\begin{aligned} &Y(\theta, \varphi + 2\pi) = Y(\theta, \varphi) , \\ &| Y(0, \varphi) | < \infty , \quad | Y(\pi, \varphi) | < \infty . \end{aligned} \quad \text{(A-3.51)}$$

The bounded solutions of Eq. (A-3.51), which possess continuous derivatives up to and including the second order, are called spherical functions (or also spherical surface functions).

The solution of the equation for $Y(\theta, \varphi)$ is also obtained by use of separation of variables, by setting

$$Y(\theta, \varphi) = \Theta(\theta)\Phi(\varphi) .$$

The function $\Phi(\varphi)$ satisfies the equation

$$\Phi'' + \mu\Phi = 0$$

and the periodicity condition

$$\Phi(\varphi + 2\pi) = \Phi(\varphi) .$$

The equation for $\Phi(\varphi)$ for integers $\mu = m^2$ has only periodic solutions, that is, $\cos m\varphi$ and $\sin m\varphi$ are linearly independent solutions. The function $\Theta(\theta)$ is determined from the equation

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) + \left(\lambda - \frac{\mu}{\sin^2\theta}\right)\Theta = 0 ,$$

where its boundedness is required for $\theta = 0$ and $\theta = \pi$. If we now introduce the substitution

$$t = \cos\theta$$

and write $X(t)|_{t=\cos\theta} = X(\cos\theta) = \Theta(\theta)$, then we obtain the equation of the associated Legendre polynomials,

$$\frac{d}{dt}\left[(1 - t^2)\frac{dX}{dt}\right] + \left(\lambda - \frac{m^2}{1 - t^2}\right)X = 0 , \qquad (-1 < t < 1) \qquad \text{(A-3.52)}$$

for $X(t)$. As we have already seen in Section A-3.1, Eq. (A-3.52) has bounded solutions only for $\lambda = n(n + 1)$. These are

$$X(t)\,|_{t=\cos\theta} = P_n^{(m)}(t)\,|_{t=\cos\theta} = P_n^{(m)}(\cos\theta) = \Theta(\theta) ,$$

where $m = n$.

We now want to enumerate the system of spherical functions of the nth order. For this we make the convention that those functions which contain $\cos k\varphi$ have a negative superscript and those functions which contain $\sin k\varphi$ have a positive superscript. Then we have

$$\begin{aligned}
&m = 0 && Y_n^{(0)} = P_n(\cos\theta) , && \\
&m = 1 && Y_n^{(-1)}(\theta, \varphi) = P_n^{(1)}(\cos\theta)\cos\varphi , && Y_n^{(1)}(\theta, \varphi) = P_n^{(1)}(\cos\theta)\sin\varphi , \\
&\cdots && \cdots && \cdots \qquad \text{(A-3.53)} \\
&m = k && Y_n^{(-k)}(\theta, \varphi) = P_n^{(k)}(\cos\theta)\cos k\varphi , && Y_n^{(k)}(\theta, \varphi) = P_n^{(k)}(\cos\theta)\sin k\varphi , \\
& && && (k = 1, 2, \cdots, n) .
\end{aligned}$$

The number of different spherical functions of nth degree $Y_n^{(m)}(\theta, \varphi)$ is, therefore, equal to $2n + 1$. A linear combination of these $2n + 1$ spherical functions (A-3.53), namely,

$$Y_n(\theta, \varphi) = \sum_{m=0}^{n}(A_{nm}\cos m\varphi + B_{nm}\sin m\varphi)P_n^{(m)}\cos(\theta) \qquad \text{(A-3.53')}$$

or

$$Y_n(\theta, \varphi) = \sum_{m=-n}^{n} C_{mn}Y_n^{(m)}(\theta, \varphi) ,$$

where

$$C_{mn} = \begin{cases} A_{nm} & \text{for } m \leqq 0 , \\ B_{nm} & \text{for } m > 0 \end{cases}$$

is again a spherical function.

The functions $Y_n^{(0)}(\theta) = P_n(\cos\theta)$ are independent of φ; they are called zonal spherical functions. Since $P_n(t)$, according to the lemma in Section A-3.1(6), has exactly n zeros within the interval $[-1, 1]$, it divides the spherical surface into $n + 1$ extended zones in which the zonal functions, as the case may be, retain their signs.

We shall now consider the behavior of the functions

$$Y_n^{(\pm k)} = \sin^k\theta \left[\frac{d^k}{dt^k} P_n(t)\right]_{t=\cos\theta} \begin{cases} \sin k\varphi , \\ \cos k\varphi \end{cases}$$

on the sphere. Since $\sin\theta$ at the poles, $\sin k\varphi$ or $\cos k\varphi$ on the $2k$ meridian circles, and $(d^k/dt)P_n(t)$, according to the lemma mentioned, are equal to zero on the $n - k$ extended circles, the entire spherical surface is divided into cells in which $Y_n^{(\pm k)}(\theta, \varphi)$ are called tesseral spherical functions (Figure 98).

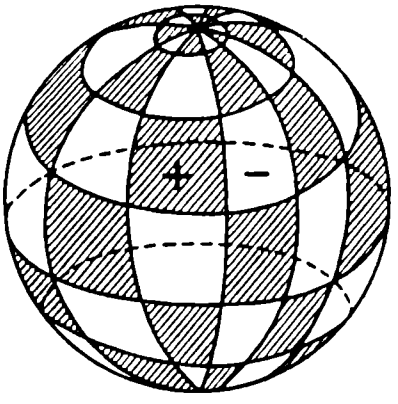


FIG. 98

We turn now to the determination of the function $R(r)$ and set

$$R = r^\sigma .$$

If we substitute this into the Euler differential equation (A-3.50), we obtain for σ

$$\sigma(\sigma + 1) - n(n + 1) = 0 ,$$

which is satisfied by the two values

$$\sigma = n \qquad \text{and} \qquad \sigma = -(n + 1) .$$

Consequently, we have as special solutions of the Laplace differential equation the functions

$$r^n Y_n^{(k)}(\theta, \varphi) , \tag{A-3.53''}$$

$$r^{-(n+1)} Y_n^{(k)}(\theta, \varphi) . \tag{A-3.53'''}$$

The first of these obviously corresponds to the solution of interior problems, while the second function corresponds to exterior problems.

We shall now show that these solutions of the Laplace differential equation can be treated in terms of a homogeneous polynomial of the nth degree. The general term, for example, in formula (A-3.53″) can be written as follows:

$$v = r^n \sin^k \theta \cos k\varphi \cos^{n-k-2q} \theta ,$$

where q varies from zero to $(n - k)/2$. The function v can be represented as the product of three polynomials:

$$v = u_1 \cdot u_2 \cdot u_3 ,$$

where

$$u_1 = r^k \sin^k \theta \cos k\varphi = \operatorname{Re}[r \sin \theta e^{i\varphi}]^k = \operatorname{Re}[(x + iy)^k] ,$$
$$u_2 = r^{n-k-2q} \cos^{n-k-2q} \theta = z^{n-k-2q} ,$$
$$u_3 = r^{2q} = (x^2 + y^2 + z^2)^q .$$

From this it is apparent that the function $r^n Y_n^{(k)}(\theta, \varphi)$ is indeed a homogeneous harmonic polynomial of degree $k + n - k - 2q + 2q = n$.

Obviously, the spherical functions represent the values of the functions (A-3.53″) and (A-3.53‴) on the unit sphere.

3. *The orthogonality of the system of spherical functions.* We now show that spherical functions which correspond to different values of λ are orthogonal to each other on the spherical surface Σ. Let Y_1 and Y_2 be two functions which satisfy the equations

$$\Delta_{\theta,\varphi} Y_1 + \lambda_1 Y_1 = 0 , \qquad \Delta_{\theta,\varphi} Y_2 + \lambda_2 Y_2 = 0 \tag{A-3.51′}$$

with

$$\Delta_{\theta,\varphi} = \frac{1}{\sin \theta} \frac{\partial}{\partial \theta}\left(\sin \theta \frac{\partial}{\partial \theta}\right) + \frac{1}{\sin^2 \theta} \frac{\partial^2}{\partial \varphi^2} .$$

As can be shown by partial integration,

$$\int_\Sigma\int Y_2 \Delta_{\theta,\varphi} Y_1 \, d\Omega = -\int_\Sigma\int \left\{\frac{\partial Y_1}{\partial \theta}\frac{\partial Y_2}{\partial \theta} + \frac{1}{\sin^2 \theta}\frac{\partial Y_1}{\partial \varphi}\cdot\frac{\partial Y_2}{\partial \varphi}\right\} d\Omega , \tag{A-3.54}$$
$$(d\Omega = \sin \theta \, d\theta \, d\varphi)$$

holds.

On the spherical surface we have

$$\operatorname{grad} u = \frac{\partial u}{\partial \theta} i_\theta + \frac{1}{\sin \theta}\frac{\partial u}{\partial \varphi} i_\varphi ,$$

$$\operatorname{div} A = \frac{1}{\sin \theta}\left[\frac{\partial}{\partial \theta}(\sin \theta A_\theta) + \frac{\partial A_\varphi}{\partial \varphi}\right] ,$$

so that

$$\Delta_{\theta,\varphi} u = \operatorname{div} \operatorname{grad} u .$$

Formula (A-3.54), therefore, can also be written in the form

$$\int_\Sigma\int Y_2 \Delta Y_1 \, d\Omega = -\int_\Sigma\int \operatorname{grad} Y_1 \cdot \operatorname{grad} Y_2 \cdot d\Omega .$$

If, in formula (A-3.54), the functions Y_1 and Y_2 are interchanged with each other and the expression thus obtained subtracted from (A-3.54), we obtain

$$J = \int_{\Sigma}\int \{Y_2 J_{\theta,\varphi} Y_1 - Y_1 J_{\theta,\varphi} Y_2\}\, d\Omega = 0 . \qquad (\Lambda\ 3.55)$$

Formulas (A-3.54) and (A 3.55) are Green's formulas for the operator of spherical functions.

From formula (A 3.55) the orthogonality of the functions Y_1 and Y_2 is easily obtained. By using Eq. (A 3.51) we obtain, namely from formula (A-3.55),

$$J = (\lambda_2 - \lambda_1)\int_{\Sigma}\int Y_1 Y_2\, d\Omega = 0 ,$$

whereas for $\lambda_1 \neq \lambda_2$

$$\int_{\Sigma}\int Y_1 Y_2\, d\Omega = 0$$

or

$$\int_0^{2\pi}\int_0^{\pi} Y_1(\theta, \varphi) Y_2(\theta, \varphi) \sin\theta\, d\theta\, d\varphi \quad 0$$

follows. Thus the orthogonality of the spherical functions which correspond to different values of λ is proved.

In the above, for $\lambda = n(n + 1)$ we obtained a system of $2n + 1$ spherical functions of the nth order. These spherical functions are also orthogonal to each other on the sphere.

Let $Y_n^{(k_1)}$ and $Y_n^{(k_2)}$ be two such spherical functions. By integration of their products, and bearing in mind formulas (A 3.7) and (A 3.26), we obtain

$$\begin{aligned}
\int_{\Sigma}\int Y_n^{(k_1)}\, Y_n^{(k_2)}\, d\Omega \quad & \int_0^{2\pi}\int_0^{\pi} Y_n^{(k_1)}(\theta, \varphi) Y_n^{(k_2)}(\theta, \varphi) \sin\theta\, d\theta\, d\varphi \\
& \int_0^{2\pi} \cos k_1\varphi \cos k_2\varphi\, d\varphi \int_0^{\pi} P_n^{(k_1)}(\cos\theta) P_n^{(k_2)}(\cos\theta) \sin\theta\, d\theta \\
= & \int_0^{2\pi} \cos k_1\varphi \cos k_2\varphi\, d\varphi \int_{-1}^{+1} P_n^{(k_1)}(t) P_n^{(k_2)}(t)\, dt \qquad (\Lambda\text{-}3.54') \\
& 0 \qquad\qquad \text{for } k_1 \neq k_2 , \\
& \frac{2\pi}{2n+1}\frac{(n+k)!}{(n-k)!} \qquad \text{for } k_1 \quad k_2 \ k \neq 0 , \\
& 2\pi\frac{2}{2n+1} \qquad\qquad \text{for } k_1 = k_2 \quad 0 .
\end{aligned}$$

In other words, the spherical functions defined by formulas (A-3.53) form an orthogonal system in the region $0 \leqq \theta \leqq \pi$, $0 \leqq \varphi \leqq 2\pi$. For their norms we obtain

$$\int_0^{2\pi}\int_0^{\pi} [Y_n^{(k)}(\theta, \varphi)]^2 \sin\theta\, d\theta\, d\varphi = \frac{2}{2n+1}\pi\varepsilon_k\frac{(n+k)!}{(n \quad k)!} \qquad (\Lambda\text{-}3.54'')$$

where $\varepsilon_0 = 2$ and $\varepsilon_k \quad 1$ for $k > 0$.

Under the assumption that the function $f(\theta, \varphi)$ can be expanded in a series of spherical functions [the possibility of such an expansion for twice continuously differentiable functions will be proved more precisely in Section A-3.2(5)] which can be integrated termwise, we obtain

$$f(\theta, \varphi) = \sum_{n=0}^{\infty} \sum_{m=0}^{n} (A_{nm} \cos m\varphi + B_{nm} \sin m\varphi) P_n^{(m)}(\cos \theta) = \sum_{n=0}^{\infty} Y_n(\theta, \varphi) ,$$

where A_{nm} and B_{nm} are Fourier coefficients defined by

$$A_{nm} = \frac{\int_0^{2\pi} \int_0^{\pi} f(\theta, \varphi) P_n^{(m)}(\cos \theta) \cos m\varphi \sin \theta \, d\theta \, d\varphi}{N_{nm}} ,$$

$$B_{nm} = \frac{\int_0^{2\pi} \int_0^{\pi} f(\theta, \varphi) P_n^{(m)}(\cos \theta) \sin m\varphi \sin \theta \, d\theta \, d\varphi}{N_{nm}} ,$$

$$N_{nm} = \frac{2\pi \varepsilon_m}{2n+1} \frac{(n+m)!}{(n-m)!} , \quad \varepsilon_m = \begin{matrix} 2 & \text{for } m = 0 , \\ 1 & \text{for } m > 0 . \end{matrix}$$

The general solution of the Laplace differential equation can be represented in the form

$$u(r, \theta, \varphi) = \sum_{n=0}^{\infty} \left(\frac{r}{a} \right)^n Y_n(\theta, \varphi)$$

for the interior boundary-value problem and in the form

$$u(r, \theta, \varphi) = \sum_{n=0}^{\infty} \left(\frac{a}{r} \right)^{n+1} Y_n(\theta, \varphi)$$

for the exterior boundary-value problem.

If the boundary condition $u|_{\Sigma} = f(\theta, \varphi)$ is given on the sphere of radius a, then we obtain

$$f(\theta, \varphi) = \sum_{n=0}^{\infty} Y_n(\theta, \varphi) .$$

4. *The completeness of the system of spherical functions.* The completeness of the system of spherical functions defined by formula (A-3.53) will now be proved. To do this we shall first prove that each twice continuously differentiable function $f(\theta, \varphi)$ can be approximated uniformly by polynomials in the spherical functions.

For this purpose we consider the Fourier series of such a function.

$$f(\theta, \varphi) = \sum_{m=0}^{\infty} [A_m(\theta) \cos m\varphi + B_m(\theta) \sin m\varphi] .$$

On the basis of the boundedness of the second derivatives, we easily obtain the estimates

$$|A_m| < \frac{M}{m^2} , \qquad |B_m| < \frac{M}{m^2} ,$$

where

$$M = \max |f_{\varphi\varphi}|$$

for the coefficients A_m and B_m of this expansion. From these estimates, for a sufficiently large $m_0 = m_0(\varepsilon')$, the estimate

$$\left| f - \sum_{m=0}^{m_0} [A_m(\theta) \cos m\varphi + B_m(\theta) \sin m\varphi] \right| = |R_{m_0}| < 2M \sum_{m_0}^{\infty} \frac{1}{m^2} < \varepsilon' \quad \text{(A-3.56)}$$

follows, where $\varepsilon' > 0$ is an arbitrarily prescribed number.

The Fourier coefficients $A_m(\theta)$ and $B_m(\theta)$ are continuous functions of θ which are equal to zero for $\theta = 0$ and $\theta = \pi$. Thus, on the basis of Section A-3.1(8), they can be approximated uniformly by linear combinations of the associated Legendre polynomials of the mth order:

$$\begin{aligned} \left| A_m(\theta) - \sum_{k=0}^{n} a_k P_k^{(m)}(\cos\theta) \right| &< \frac{\varepsilon'}{2m_0 + 1}, \\ \left| B_m(\theta) - \sum_{k=0}^{n} b_k P_k^{(m)}(\cos\theta) \right| &< \frac{\varepsilon'}{2m_0 + 1}. \end{aligned} \quad \text{(A-3.57)}$$

From the inequalities (A-3.56) and (A-3.57), however, we obtain

$$\left| f(\theta, \varphi) - \sum_{m=0}^{m_0} \sum_{k=0}^{n} [a_k P_k^{(m)}(\cos\theta) \cos m\varphi + b_k P_k^{(m)}(\cos\theta) \sin m\varphi] \right| < 2\varepsilon', \quad \text{(A-3.58)}$$

whereby the possibility of the uniform approximation of every twice continuously differentiable function $f(\theta, \varphi)$ by polynomials in spherical functions is proved. From this it also follows that every continuous function can be uniformly approximated and, therefore, also approximated in the mean by polynomials of spherical functions. Consequently, the system of functions (A-3.53) is complete. From the completeness follows the closedness of the system.

We have therefore shown that the equation for spherical functions for $\lambda \neq n(n+1)$ possesses no bounded solutions and that every spherical function of the nth order ($\lambda = n(n+1)$) is representable by formula (A-3.53′).

5. *The expansion in spherical functions.* Spherical functions are the bounded eigenfunctions of the equations

$$\frac{1}{\sin\theta} \frac{\partial}{\partial\theta} \left(\sin\theta \frac{\partial u}{\partial\theta} \right) + \frac{1}{\sin^2\theta} \frac{\partial^2 u}{\partial\varphi^2} + \lambda u = 0 \quad \text{or} \quad \Delta_{\theta,\varphi} u + \lambda u = 0 \quad \text{(A-3.59)}$$

on the surface of the sphere $\Sigma(0 \leqq \varphi \leqq 2\pi,\ 0 \leqq \theta \leqq \pi)$. In order to prove that each twice continuously differentiable function $f(\theta, \varphi)$ can be expanded in a series of spherical functions, we make use of the corresponding integral equation. For this purpose we construct the Green's function of the equation

$$\Delta_{\theta,\varphi} u = \frac{1}{\sin\theta} \frac{\partial}{\partial\theta} \left(\sin\theta \frac{\partial u}{\partial\theta} \right) + \frac{1}{\sin^2\theta} \frac{\partial^2 u}{\partial\varphi^2} = 0, \quad \text{(A-3.60)}$$

with the additional requirement that the solution for $\theta = 0$ and $\theta = \pi$ is to be bounded.

As already noted above, on the spherical surface we have

$$\Delta_{\theta,\varphi} u = (\operatorname{div} \operatorname{grad} u)_{\theta,\varphi}. \quad \text{(A-3.61)}$$

Equation (A-3.60) can be regarded as the equation of a stationary temperature distribution or a stationary electrical current on the spherical surface.

From this viewpoint, it is also clear that no solution of the homogeneous equation

$$\Delta_{\theta,\varphi} u = 0 \tag{A-3.62}$$

can be constructed, showing a singularity only at one point. For the existence of a stationary temperature distribution, therefore, it is necessary that the sum of all sources and sinks equal zero.

For this reason we introduce a generalized Green's function. In our case, this is defined as the solution of the equation

$$\Delta_{\theta,\varphi} u = q\,, \qquad \left(q = \frac{1}{4\pi}\right), \tag{A-3.63}$$

which is regular everywhere with the exception of the pole $\theta = 0$; there it possesses a logarithmic singularity. The right side of Eq. (A-3.63) gives the density of the negative sources (sinks) of heat which are uniformly distributed on the spherical surface, so that

$$\int_{\Sigma}\int q \, d\sigma = 1\,. \tag{A-3.64}$$

With the assumption that the Green's function sought depends only on the variable θ, we obtain for u an ordinary differential equation with the solution

$$u = -q \ln \sin\theta + c \ln \operatorname{tg}\frac{\theta}{2}\,. \tag{A-3.65}$$

Now, if u possesses a singularity only for $\theta = 0$, then necessarily

$$c = -q\,,$$

and consequently,

$$u = -2q \ln \sin\frac{\theta}{2} - q \ln 2\,.$$

Since $u_1 = \text{const}$ solves the homogeneous equation, Green's function G is determined only to within an arbitrary constant. Therefore, we can write:

$$G = -\frac{1}{2\pi} \ln \sin\frac{\theta}{2}\,. \tag{A-3.66}$$

If the source lies at the point M_0, then Green's function has the form

$$G(M, M_0) = -\frac{1}{2\pi} \ln \sin\frac{\gamma_{MM_0}}{2}\,, \tag{A-3.67}$$

where γ_{MM_0} is the angle between the points $M_0(\theta_0, \varphi_0)$ and $M(\theta, \varphi)$.[57]

We turn now to the solution of the inhomogeneous equation

$$\Delta_{\theta,\varphi} u \equiv \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial u}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2 u}{\partial\varphi^2} = -F(\theta, \varphi)\,. \tag{A-3.68}$$

[57] The angle γ is calculated from the formula

$$\cos\gamma = \cos\theta\cos\theta_0 + \sin\theta\sin\theta_0\cos(\varphi - \varphi_0)\,.$$

This equation can possess only a regular solution everywhere on Σ, if the condition

$$\int_\Sigma\int F\,d\sigma = 0 \tag{A-3.69}$$

is satisfied. This condition says that the sum of the sources and sinks must be equal to zero. We obtain it easily from Green's formulas established in Section A-3.2 (3) for the operator $\Delta_{\theta,\varphi}$.

Now we shall show that each solution of Eq. (A-3.68) which satisfies the condition (A-3.69) can be represented in the form

$$u(M) = \int_\Sigma\int G(M, P)F(P)\,d\sigma_P + A\ .$$

Here A denotes a fixed constant and $G(M, P)$ is the Green's function defined by Eq. (A 3.67). Let M be a fixed point on the sphere which we choose as the north pole ($\theta = 0$) and let M_1 be its diametrically opposite point. The points M and M_1 are then singular points of Eq. (A-3.68). We surround these points with small circles K_ε^M, $K_\varepsilon^{M_1}$ and consider the integral

$$I = \iint_{\Sigma_1=\Sigma-K_\varepsilon^M-K_\varepsilon^{M_1}} (u\Delta G - G\Delta u)\,d\sigma\ .$$

By substitution of the expression for Δu and ΔG into the right side, we obtain

$$I = \int_0^{2\pi}\int_\varepsilon^{\pi-\varepsilon}\left[u\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial G}{\partial\theta}\right) - G\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial u}{\partial\theta}\right)\right]d\theta\,d\varphi$$
$$+\int_\varepsilon^{\pi-\varepsilon}\frac{d\theta}{\sin\theta}\int_0^{2\pi}\left[u\frac{\partial^2 G}{\partial\varphi^2} - G\frac{\partial^2 u}{\partial\varphi^2}\right]d\varphi\ .$$

Now, if we bear in mind that in the square brackets the derivatives of the expressions

$$\sin\theta\left[u\frac{\partial G}{\partial\theta} - G\frac{\partial u}{\partial\theta}\right] \quad \text{and} \quad u\frac{\partial G}{\partial\varphi} - G\frac{\partial u}{\partial\varphi}$$

occur, then we obtain by integration

$$I = \int_0^{2\pi}\left[\sin\theta\left(u\frac{\partial G}{\partial\theta} - G\frac{\partial u}{\partial\theta}\right)\right]_\varepsilon^{\pi-\varepsilon} d\varphi\ .$$

Furthermore, because of

$$\frac{\partial G}{\partial\theta} = -\frac{1}{2\pi}\frac{\partial}{\partial\theta}\ln\sin\frac{\theta}{2} = -\frac{1}{4\pi}\operatorname{ctg}\frac{\theta}{2}\ ,$$

there results

$$I = \frac{1}{2\pi}\int_0^{2\pi}\left[\sin\frac{\theta}{2}\cos\frac{\theta}{2}\operatorname{ctg}\frac{\theta}{2}\cdot u\right]_\varepsilon^{\pi-\varepsilon} d\varphi$$
$$-\frac{1}{2\pi}\left[\sin\theta\ln\sin\frac{\theta}{2}\int_0^{2\pi}\frac{\partial u}{\partial\theta}\,d\varphi\right]_\varepsilon^{\pi-\varepsilon} = I_1 + I_2\ .$$

From this it is apparent that

$$\lim_{\varepsilon \to 0} I_1 = u(M) \qquad \text{and} \qquad \lim_{\varepsilon \to 0} I_2 = 0 .$$

Consequently,

$$u(M) = \int_{\Sigma}\int G(M, P)F(P)\, d\sigma_P + A , \tag{A-3.70}$$

where

$$A = \frac{1}{4\pi}\int_{\Sigma}\int u\, d\sigma$$

is constant. The solution of our problem is defined only up to an additive constant. Thus, when

$$\int_{\Sigma}\int u\, d\sigma = 0 ,$$

the solution is given by the formula

$$u(M) = \int_{\Sigma}\int G(M, P)F(P)\, d\sigma_P .$$

By application of Eq. (A-3.70) to the equation for spherical functions $J_{\theta,\varphi}u = -\lambda u$, we arrive at the following conclusions.

The spherical functions defined by Eq. (A-3.53) represent the totality of all linear independent eigenfunctions of the integral equation

$$u(M) = \lambda \int_{\Sigma}\int G(M, P)u(P)\, d\sigma_P$$

with symmetrical kernel $G(M, P)$ defined by Eq. (A-3.67).

The general theory of integral equations with a symmetrical kernel is applicable to this equation. From this theory it follows that each twice continuously differentiable function $f(\theta, \varphi)$ can be expanded in a uniformly and absolutely convergent series of spherical functions:

$$f(\theta, \varphi) = \sum_{n=0}^{\infty} Y_n(\theta, \varphi) = \sum_{n=0}^{\infty}\sum_{m=0}^{n} (A_{nm}\cos m\varphi + B_{nm}\sin m\varphi)P_n^{(m)}(\cos\theta) , \tag{A-3.71}$$

where

$$Y_n(\theta, \varphi) = \sum_{m=0}^{n} (A_{nm}\cos m\varphi + B_{nm}\sin m\varphi)P_n^{(m)}(\cos\theta) \tag{A-3.72}$$

and $A_{n,m}$, $B_{n,m}$ denote the Fourier coefficients.

3. Some examples of the application of spherical functions

1. *The polarization of a sphere in a homogeneous field.* As an example of the application of spherical functions, we shall treat the problem of the polarization of a dielectric sphere in a homogeneous field.

Consider an electrostatic field in a homogeneous isotropic medium with the dielectric constant ε_1. In this medium we place a dielectric sphere of

radius a with dielectric constant ϵ_2 (Figure 99). We seek the potential of the resulting field in the form

$$u = \begin{cases} u_1 = u_0 + v_1 & \text{outside of the sphere} \\ u_2 = u_0 + v_2 & \text{inside of the sphere.} \end{cases}$$

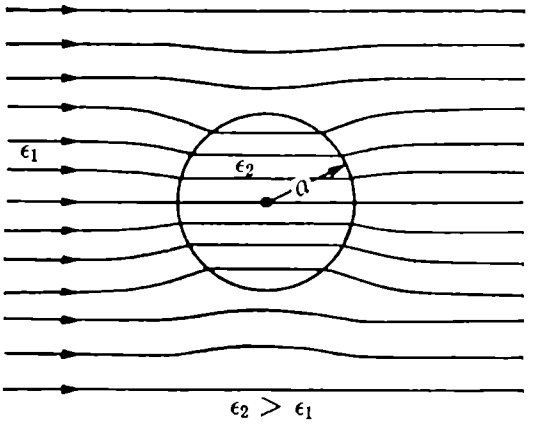


FIG. 99

Here u_0 denotes the potential of the field which would be present in the absence of the dielectric sphere, and v is the perturbation of the field caused by the presence of the sphere. The potential u satisfies the equation

$$\Delta u = 0$$

with the additional conditions

$$u_1 = u_2 \quad \text{on } S$$

$$\epsilon_1 \frac{\partial u_1}{\partial n} = \epsilon_2 \frac{\partial u_2}{\partial n} \quad \text{on } S .$$

The boundary of the sphere is S; the values of u outside and inside the sphere are u_1 and u_2, respectively. From this it follows that the potential v can be determined from the requirements

$$\Delta v = 0 , \tag{A-3.73}$$

$$v_1 = v_2 \quad \text{on } S , \tag{A-3.74}$$

$$\epsilon_1 \frac{\partial v_1}{\partial n} - \epsilon_2 \frac{\partial v_2}{\partial n} = -(\epsilon_1 - \epsilon_2) \frac{\partial u_0}{\partial n} \quad \text{on } S , \tag{A-3.75}$$

since for the function u_0, we have

$$\Delta u_0 = 0 , \qquad (u_0)_1 = (u_0)_2 \quad \text{on } S ,$$

$$\left(\frac{\partial u_0}{\partial n}\right)_1 = \left(\frac{\partial u_0}{\partial n}\right)_2 \quad \text{on } S .$$

On the right side of the Eq. (A-3.75) is a known function of θ and φ, which we expand in spherical functions

$$\left.\frac{\partial u_0}{\partial n}\right|_S = \sum_{n=0}^{\infty} Y_n(\theta, \varphi) .$$

If we now set

$$v_1 = \sum_{n=0}^{\infty} \left(\frac{a}{r}\right)^{n+1} \bar{Y}_n(\theta, \varphi) , \qquad v_2 = \sum_{n=0}^{\infty} \left(\frac{r}{a}\right)^{n} \bar{\bar{Y}}_n ,$$

then, by consideration of the boundary conditions (A-3.74) and (A-3.75), we obtain

$$\bar{Y}_n = \bar{\bar{Y}}_n ,$$

and

$$\varepsilon_1 \sum_{n=0}^{\infty} \frac{-(n+1)}{r} \left(\frac{a}{r}\right)^{n+1} \bar{Y}_n \Big|_{r=a} - \varepsilon_2 \sum_{n=0}^{\infty} \frac{n}{a} \left(\frac{r}{a}\right)^{n-1} \bar{\bar{Y}}_n \Big|_{r=a} = -(\varepsilon_1 - \varepsilon_2) \sum_{n=0}^{\infty} Y_n ,$$

from which

$$\bar{Y}_n = Y_n \frac{(\varepsilon_1 - \varepsilon_2)a}{\varepsilon_1(n+1) + \varepsilon_2 n} \tag{A-3.76}$$

follows.

We shall consider a special case. The sphere is immersed in a homogeneous parallel external field E_0, which is parallel to the z-axis. The potential of this field is equal to

$$u_0 = -E_0 z = -E_0 r \cos\theta ,$$

so that

$$\frac{\partial u_0}{\partial n}\Big|_S = \frac{\partial u_0}{\partial r}\Big|_{r=a} = -E_0 \cos\theta = Y_1(\theta) .$$

From Eq. (A-3.76) we obtain

$$\bar{Y}_n = 0 \quad \text{for } n \neq 1 ,$$

$$\bar{Y}_1 = -E_0 \cos\theta \frac{(\varepsilon_1 - \varepsilon_2)a}{2\varepsilon_1 + \varepsilon_2} .$$

For the potential of the resulting field in the presence of the sphere,

$$u_1 = -E_0 z \left[1 + \frac{\varepsilon_1 - \varepsilon_2}{2\varepsilon_1 + \varepsilon_2} \left(\frac{a}{r}\right)^3\right] \quad \text{outside of the sphere,} \quad (r > a),$$

$$u_2 = -E_0 z \frac{3\varepsilon_1}{2\varepsilon_1 + \varepsilon_2} \quad \text{inside of the sphere,} \quad (r < a),$$

from which

$$E_1 = -\frac{\partial u_1}{\partial z} = \left[1 - \frac{\varepsilon_1 - \varepsilon_2}{2\varepsilon_1 + \varepsilon_2} \frac{2a^3}{r^3}\right] E_0 ,$$

$$E_2 = -\frac{\partial u_2}{\partial z} = \frac{3\varepsilon_1}{2\varepsilon_1 + \varepsilon_2} E_0$$

follows, that is, the field inside the sphere is parallel and homogeneous.

If $\varepsilon_2 > \varepsilon_1$, then the equipotential surfaces are represented by planes which are perpendicular to the direction of the field and lie further apart than in the original field without the sphere. The lines of force, which are the orthogonal trajectories of the equipotential planes, are imbedded in the sphere with the larger dielectric constant. In the case $\varepsilon_1 > \varepsilon_2$ the picture is reversed.

In a similar manner, the problem of polarization of a sphere in the presence of a point source can be solved if the expansion of $1/r$ in spherical functions is employed (see Section A-3.1).

Let us recall that the corresponding problems in the investigation of magnetic and thermal fields, as well as for fields of stationary electrical currents, occur when the existing inhomogeneity has the form of a sphere. For thermal problems, in the boundary condition (A-3.75), in place of ε_1 and ε_2 we have the heat conductivity coefficients k_1 and k_2; in the case of a magnetic field, the magnetic permeabilities μ_1 and μ_2, and, finally, in the case of a field of a stationary current, the conductivities λ_1 and λ_2.

2. *The eigenvibrations of a sphere.* The eigenvibrations of a sphere of radius r_0 with homogeneous boundary conditions of the first kind will now be determined. This problem leads to the determination of the eigenvalues and eigenfunctions of the equation

$$\Delta v + \lambda v = 0 \tag{A-3.77}$$

with the boundary condition

$$v = 0 \tag{A-3.78}$$

on the surface of sphere. If we introduce polar coordinates and place the origin at the center of the sphere, then Eq. (A-3.77) has the form

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial v}{\partial r}\right) + \frac{1}{r^2}\Delta_{\theta,\varphi}v + \lambda v = 0\,, \tag{A-3.77'}$$

where

$$\Delta_{\theta,\varphi}v = \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial v}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2 v}{\partial\varphi^2}\,.$$

The solution of this equation is determined by separation of variables in which we set

$$v(r,\theta,\varphi) = R(r)Y(\theta,\varphi)\,. \tag{A-3.78'}$$

If we put this into Eq. (A-3.77), then we obtain

$$\frac{\dfrac{d}{dr}\left(r^2\dfrac{dR}{dr}\right)}{R} + \lambda r^2 + \frac{\Delta_{\theta,\varphi}Y}{Y} = 0\,, \tag{A-3.79}$$

from which

$$\Delta_{\theta,\varphi}Y + \mu Y = 0\,, \tag{A-3.80}$$

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) + \left(\lambda - \frac{\mu}{r^2}\right)R = 0 \tag{A-3.81}$$

follow.

For the solutions of Eq. (A-3.80) we naturally require that they are bounded at the poles of the sphere

$$|Y|_{\theta=0,\pi} < \infty \tag{A-3.82}$$

and possess the period 2π with respect to φ: $Y(\theta, \varphi) = Y(\theta, \varphi + 2\pi)$. These requirements are satisfied by the eigenfunctions corresponding to the eigenvalues

$$\mu = n(n+1) \tag{A-3.83}$$

and, in fact, to each of these eigenvalues belongs the $2n+1$ spherical functions

$$\begin{aligned} Y_n^{(-j)}(\theta, \varphi) &= P_n^{(j)}(\cos\theta)\cos j\varphi \,, \\ Y_n^{(j)}(\theta, \varphi) &= P_n^{(j)}(\cos\theta)\sin j\varphi \,, \end{aligned} \qquad (j = 0, 1, 2, \ldots, n) \,. \tag{A-3.84}$$

We turn now to Eq. (A-3.81). By consideration of Eq. (A-3.83), the boundary condition for $r = r_0$, and the boundedness condition for $r = 0$, we obtain the eigenvalue problem for $R(r)$

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) + \left(\lambda - \frac{n(n+1)}{r^2}\right)R = 0 \,, \tag{A-3.81'}$$

$$R(r_0) = 0 \,, \tag{A-3.85}$$

$$|R(0)| < \infty \,. \tag{A-3.86}$$

With the help of the substitution

$$R(r) = \frac{y(r)}{\sqrt{r}} \,, \tag{A-3.87}$$

Eq. (A-3.81′) leads to the Bessel equation of order $(n + \frac{1}{2})$

$$y'' + \frac{1}{r}y' + \left[\lambda - \frac{(n+\frac{1}{2})^2}{r^2}\right]y = 0 \,, \tag{A-3.88}$$

whose general solution has the form

$$y(r) = AJ_{n+1/2}(\sqrt{\lambda}\, r) + BN_{n+1/2}(\sqrt{\lambda}\, r) \tag{A-3.89}$$

[see Section A-2.1 (1)]. From the boundedness requirement (A-3.86), there now follows

$$B = 0 \,.$$

Furthermore, the boundary condition (A-3.85) gives

$$AJ_{n+1/2}(\sqrt{\lambda}\, r_0) = 0 \,.$$

Since only the nontrivial solutions are of interest to us, $A \neq 0$ and, consequently,

$$J_{n+1/2}(\sqrt{\lambda}\, r_0) = 0.$$

Now let $\nu_1^{(n)}, \nu_2^{(n)}, \ldots, \nu_m^{(n)}, \ldots$ be the roots of the transcendental equation

$$J_{n+1/2}(\nu) = 0 \,. \tag{A-3.90}$$

Then

$$\lambda_{m,n} = \left(\frac{\nu_m^{(n)}}{r_0}\right)^2 \tag{A-3.91}$$

are the eigenvalues sought. Each eigenvalue $\lambda_{m,n}$ corresponds to $2n+1$ eigenfunctions. We introduce the expression

$$\psi_n(x) = \sqrt{\frac{\pi}{2x}}\, J_{n+1/2}(x) \,. \tag{A-3.92}$$

Then the eigenfunctions of Eq. (A-3.77) with the boundary condition (A-3.78) can be represented in the following manner:

$$v_{n,m,j}(r, \theta, \varphi) = \psi_n\left(\frac{\nu_m^{(n)}}{r_0} r\right) Y_n^{(j)}(\theta, \varphi) ,$$

$$(j = -n, \ldots, -1, 0, 1, \ldots, n) . \tag{A-3.93}$$

We shall now treat the first interior boundary-value problem for the vibration equation

$$\Delta v + k^2 v = 0 \tag{A-3.94}$$

with the boundary condition

$$v = f(\theta, \varphi) \tag{A-3.95}$$

on the surface of a sphere of radius r_0.

From the above investigation we infer that the solution of this problem can be represented in the form

$$v(r, \theta, \varphi) = \sum_{n=0}^{\infty} \sum_{j=-n}^{\infty} f_{nj} \frac{\psi_n(kr)}{\psi_n(kr_0)} Y_n^{(j)}(\theta, \varphi) , \tag{A-3.96}$$

where f_{nj} are the coefficients in the expansion of $f(\theta, \varphi)$ in spherical functions

$$f(\theta, \varphi) = \sum_{n=0}^{\infty} \sum_{j=-n}^{n} f_{nj} Y_n^{(j)}(\theta, \varphi) . \tag{A-3.97}$$

If k^2 coincides with one of the eigenvalues

$$k^2 = \lambda_{m_0 n_0} = \left(\frac{\nu_{m_0}^{(n_0)}}{r_0}\right)^2 ,$$

then the problem (A-3.94)-(A-3.95) does not have a solution for every function $f(\theta, \varphi)$. Formula (A-3.96) shows that our problem has exactly one solution if the coefficients $f_{n_0 j}$ are equal to zero:

$$f_{n_0 j} = 0$$

or

$$\int_0^{\pi} \int_0^{2\pi} f(\theta, \varphi) Y_{n_0}^{(j)}(\theta, \varphi) \sin \theta \, d\theta \, d\varphi = 0 .$$

If these conditions are satisfied, then the solution is given by formula (A-3.96) in which the missing summands correspond to the index $n = n_0$. However, here the solution is not uniquely determined since it still can include an arbitrary linear combination of eigenfunctions which correspond to the eigenvalue $k^2 = \lambda_{m_0 n_0}$.

3. *The exterior boundary-value problem for the sphere.* The exterior first boundary-value problem for the sphere reads

$$\Delta v + k^2 v = 0 , \qquad (k^2 > 0) ,$$

$$v|_{r=r_0} = f(\theta, \varphi) ,$$

$$v = O\left(\frac{1}{r}\right) \quad \text{for } r \to \infty$$

$$\lim_{r\to\infty} r\left(\frac{\partial v}{\partial r} + ikv\right) = 0 , \quad \text{(radiation condition).}$$

As was shown in Section 7-3, this problem possesses a unique solution. We first expand the function sought and $f(\theta, \varphi)$ in a series of spherical functions

$$v(r, \theta, \varphi) = \sum_{n=0}^{\infty} \sum_{j=-n}^{n} R_n(r) Y_n^{(j)}(\theta, \varphi) ,$$

$$f(\theta, \varphi) = \sum_{n=0}^{\infty} \sum_{j=-n}^{n} f_{nj} Y_n^{(j)}(\theta, \varphi) .$$

The expansion coefficients $R_n(r)$ obviously satisfy the equation

$$R_n'' + \frac{1}{r} R_n' + \left(k^2 - \frac{n(n+1)}{r^2}\right) R_n = 0 ,$$

the boundary condition

$$R_n(r_0) = f_n ,$$

and, further, the radiation conditions for $r \to \infty$:

$$R_n(r) = O\left(\frac{1}{r}\right) ,$$

$$\lim_{r\to\infty} r(R_n' + ikR_n) = 0 .$$

The general solution of this equation has the form [see Section A-3.3 (2) and Vol. I, Appendix, p. 364]

$$R_n(r) = A_n \zeta_n^{(1)}(kr) + B_n \zeta_n^{(2)}(kr) ,$$

where

$$\zeta_n^{(1)}(\rho) = \sqrt{\frac{\pi}{2\rho}} H_{n+1/2}^{(1)}(\rho) ,$$

$$\zeta_n^{(2)}(\rho) = \sqrt{\frac{\pi}{2\rho}} H_{n+1/2}^{(2)}(\rho) , \qquad (\rho = kr) .$$

Now if we use the asymptotic representation of the Hankel functions $H_n^{(1)}(\rho)$ and $H_n^{(2)}(\rho)$ (see Vol. I, Appendix, p. 364)

$$H_n^{(1)}(\rho) = \sqrt{\frac{2}{\pi\rho}} e^{i[\rho - (\pi n/2) - (\pi/4)]} + \cdots ,$$

$$H_n^{(2)}(\rho) = \sqrt{\frac{2}{\pi\rho}} e^{-i[\rho - (\pi n/2) - (\pi/4)]} + \cdots$$

(the dots denote those terms of higher order which are smaller than $1/\rho$), then for the functions $\zeta_n^{(1)}$ and $\zeta_n^{(2)}$ we obtain the asymptotic representations

$$\zeta_n^{(1)}(kr) = \frac{e^{i[kr - (\pi n/2) - (\pi/4)]}}{r} + \cdots ,$$

$$\zeta_n^{(2)}(kr) = \frac{e^{-i[kr - (\pi n/2) - (\pi/4)]}}{r} + \cdots .$$

As we see from these representations, only $\zeta_n^{(2)}$ satisfies the radiation condition. Thus

$$A_n = 0 .$$

From the boundary condition for $r = r_0$ we obtain

$$B_{nj} = \frac{f_{nj}}{\zeta_n^{(2)}(kr_0)} .$$

From this we obtain the function $v(r, \theta, \varphi)$ in the form

$$v(r, \theta, \varphi) = \sum_{n=0}^{\infty} \sum_{j=-n}^{n} \frac{f_{nj}\zeta_n^{(2)}(kr)}{\zeta_n^{(2)}(kr_0)} Y_n^{j)}(\theta, \varphi) ,$$

where

$$f_{nj} = \frac{\int_0^{\pi}\int_0^{2\pi} f(\theta, \varphi) Y_n^{(j)}(\theta, \varphi) \sin\theta \, d\theta \, d\varphi}{N_{nj}}$$

and

$$N_{nj} = \int_0^{\pi}\int_0^{2\pi} [Y_n^{(j)}]^2 \sin\theta \, d\theta \, d\varphi$$
$$= \frac{2\pi\varepsilon_j}{2n+1} \frac{(n+j)!}{(n-j)!} , \qquad \varepsilon_j = \begin{cases} 2, & j = 0 , \\ 1, & j > 0 \end{cases}$$

are the norms of the spherical function $Y_n^{(j)}(\theta, \varphi)$.

A-4. THE TSCHEBYSCHEFF-HERMITE AND THE TSCHEBYSCHEFF-LAGUERRE POLYNOMIALS

1. The Tschebyscheff-Hermite polynomial

For the treatment of the linear harmonic oscillator in quantum mechanics, we arrive at the equation

$$\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + \lambda y = 0 , \tag{A-4.1}$$

which can also be written in the form

$$\frac{d}{dx}\left[e^{-x^2}\frac{dy}{dx}\right] + \lambda e^{-x^2} y = 0 . \tag{A-4.2}$$

The Tschebyscheff-Hermite polynomials are defined as solutions of eigenvalue problems for Eq. (A-4.2) on the unbounded straight line $-\infty < x < \infty$ with the following boundary condition: The solution, as $x \to \infty$, is to be infinite only as the same order as a finite power of x.

We shall seek the solution in a power series:

$$y = \sum_{n=0}^{\infty} a_n x^n . \tag{A-4.3}$$

If we put series (A-4.3) into Eq. (A-4.1), we obtain

$$\sum_{n=0}^{\infty} \{a_{n+2}(n+2)(n+1) - 2na_n + \lambda a_n\} x^n = 0 .$$

From this we obtain, for the coefficients, the recursion formula

$$a_{n+2} = \frac{2n - \lambda}{(n+2)(n+1)} a_n \,. \tag{A-4.4}$$

From formula (A-4.4) it follows that series (A-4.3) for $a_1 = 0$ contains only even powers of x and for $a_0 = 0$ only odd powers of x. The coefficients a_0 and a_1 are arbitrary constants which determine the special solutions of Eq. (A-4.1).

For $\lambda = 2n$, Eq. (A-4.1) is solved by the polynomial of the nth degree

$$\bar{H}_n(x) = a_0 + a_2 x^2 + \cdots + a_n x^n \,, \qquad (n\text{-even}),$$

or (A-4.5)

$$\bar{H}_n(x) = a_1 x + a_3 x^3 + \cdots + a_n x^n \,, \qquad (n\text{-odd}).$$

Except for the normalizing factors, these are indeed the Tschebyscheff-Hermite polynomials. They represent eigenfunctions of the boundary-value problem considered.

If $\lambda \neq 2n$ (that is, if λ is not an even nonnegative integer), then the power series possesses infinitely many coefficients different from zero which, with the possible exception of finitely many terms, have the same sign. In this case, there exists a term $a_n x^n$ with sufficiently large n which for $x \to \infty$ becomes greater than an arbitrarily prescribed power of x. Accordingly, the power series does not satisfy the boundary conditions posed and, therefore, cannot be a solution of our boundary-value problem. Thus we have shown that the Tschebyscheff-Hermite polynomials are unique solutions of the problem.

From the definition of the Tschebyscheff-Hermite polynomials $H_n(x)$, we are led to introduce the generating function

$$\Psi(x, t) = e^{2tx - t^2} \tag{A-4.6}$$

and its expansion in a Taylor series in the variable t:

$$\Psi(x, t) = e^{x^2} e^{-(t-x)^2} = \sum_{n=0}^{\infty} H_n(x) \frac{t^n}{n!} \,. \tag{A-4.7}$$

We see immediately that the expansion coefficients $H_n(x)$ are polynomials of the nth degree.

We shall now show that the polynomial $H_n(x)$ coincides with the polynomial $\bar{H}_n(x)$ defined by Eq. (A-4.5) except for a proportionality factor. From the relations

$$\frac{\partial \Psi}{\partial x} = 2t\Psi$$

and

$$\frac{\partial \Psi}{\partial t} + 2(t - x)\Psi = 0 \,,$$

we obtain

$$H_n'(x) = 2nH_{n-1}(x) \tag{A-4.8}$$

and, correspondingly,

$$H_{n+1} - 2xH_n + 2nH_{n-1} = 0 . \tag{A-4.9}$$

Furthermore, from Eqs. (A-4.8) and (A-4.9) we find that $H_n(x)$ satisfies the equation

$$H_n'' - 2xH_n' + 2nH_n = 0 .$$

In other words, the Tschebyscheff-Hermite polynomial $H_n(x)$ coincides with the polynomial $H_n(x)$ except for a proportionality factor. This remains undetermined in Eq. (A-4.5).

From Eq. (A-4.7) the following differential formula for the Tschebyscheff-Hermite polynomials is obtained:

$$H_n(x) = \left(\frac{\partial^n \Psi}{\partial t^n}\right)_{t=0} = (-1)^n e^{x^2} \frac{d^n}{dx^n} e^{-x^2} . \tag{A-4.10}$$

In particular, we find $a_n = 2^n$.

By using this formula, we obtain for the first five polynomials $H_n(x)$:

$$\begin{aligned} H_0(x) &= 1 , \\ H_1(x) &= 2x , \\ H_2(x) &= 4x^2 - 2 , \\ H_3(x) &= 8x^3 - 12x , \\ H_4(x) &= 16x^4 - 48x^2 - 12 . \end{aligned}$$

The Tschebyscheff-Hermite polynomials form, as we shall show, an orthogonal system with density e^{-x^2} in the interval $-\infty < x < \infty$:

$$\int_{-\infty}^{\infty} H_m(x) H_n(x) e^{-x^2} dx = \begin{cases} 0 & \text{for } m \neq n \\ 2^n n! \sqrt{\pi} & \text{for } m = n . \end{cases} \tag{A-4.11}$$

Thus,

$$J = \int_{-\infty}^{\infty} H_m(x) H_n(x) e^{-x^2} dx = (-1)^n \int_{-\infty}^{\infty} H_m(x) \frac{d^n}{dx^n} (e^{-x^2}) dx .$$

We take $m = n$. By partial integration and by consideration of both formula (A-4.8) and the fact that for $x \to \infty$ the product of each polynomial with e^{-x^2} and all the derivatives of arbitrarily higher order vanish, we obtain

$$\begin{aligned} J &= (-1)^{n-1} 2m \int_{-\infty}^{\infty} H_{m-1}(x) \frac{d^{n-1}}{dx^{n-1}} (e^{-x^2}) dx = \cdots \\ &= (-1)^{n-m} 2^m m! \int_{-\infty}^{\infty} \frac{d^{n-m}}{dx^{n-m}} (e^{-x^2}) dx . \end{aligned}$$

If $m < n$, then

$$J = (-1)^{n-m} 2^m m! \frac{d^{n-m-1}}{dx^{n-m-1}} (e^{-x^2}) \Big|_{-\infty}^{\infty} = 0 .$$

If $m = n$, then

$$J = 2^n n! \int_{-\infty}^{\infty} e^{-x^2} dx = 2^n n! \sqrt{\pi} ,$$

since

$$\int_{-\infty}^{\infty} e^{-x^2}\,dx = \sqrt{\pi}\ .$$

Thus the relations (A-4.11) are proved.

In applications, the functions

$$\psi_n(x) = \frac{H_n(x)e^{-x^2/2}}{\sqrt{2^n n!}\sqrt{\pi}}$$

are often used. These form a normalized orthogonal system in the interval $-\infty < x < \infty$. They satisfy the equation

$$\psi_n'' + [(2n+1) - x^2]\psi_n = 0\ .$$

2. The Tschebyscheff-Laguerre polynomials

1. The Tschebyscheff-Laguerre polynomials can be defined as solutions of the equation

$$xy'' + (1-x)y' + \lambda y = 0\ , \qquad (0 < x < \infty)\ , \tag{A-4.12}$$

or in the self-adjoint form

$$(xe^{-x}y')' + \lambda e^{-x}y = 0\ , \qquad (0 < x < \infty)\ , \tag{A-4.12'}$$

with the following boundary conditions: The solutions, bounded for $x = 0$ and for $x \to \infty$, do not become greater at infinity than a finite power of x.

We form the solutions of Eq. (A-4.12) as a power series in x:

$$y = \sum_{n=0}^{\infty} a_n x^n\ . \tag{A-4.13}$$

If we introduce Eq. (A-4.13) into Eq. (A-4.12), we find

$$\sum_{n=0}^{\infty} [a_{n+1}(n+1)n + a_{n+1}(n+1) - na_n + \lambda a_n]x^n = 0$$

or

$$\sum_{n=0}^{\infty} [a_{n+1}(n+1)^2 - a_n(n-\lambda)]x^n = 0\ ,$$

from which, for the coefficients of series (A-4.13), we obtain the recursion formula

$$a_{n+1} = \frac{n-\lambda}{(n+1)^2}\,a_n\ . \tag{A-4.14}$$

The coefficient a_0 can be chosen arbitrarily. It corresponds to a special solution of Eq. (A-4.12). For $\lambda = n$, Eq. (A-4.12) has a polynomial of the nth degree as a solution. If a_0 is so chosen that the coefficient of the highest power x^n is equal to $(-1)^n$, then we obtain directly the Tschebyscheff-Laguerre polynomial $L_n(x)$. Thus $a_0 = n!$.

The first five polynomials $L_n(x)$ read

$$L_0(x) = 1 ,$$
$$L_1(x) = -x + 1 ,$$
$$L_2(x) = x^2 - 4x + 2 ,$$
$$L_3(x) = -x^3 + 9x^2 - 18x + 6 ,$$
$$L_4(x) = x^4 - 16x^3 + 72x^2 - 96x + 24 .$$

The Tschebyscheff-Laguerre polynomial has the generating function

$$\Psi(x, t) = \frac{e^{-xt/(1-t)}}{1-t} . \tag{A-4.15}$$

If this is expanded in powers of t, then we obtain

$$\Psi(x, t) = \frac{e^{-xt/(1-t)}}{1-t} = \sum_{n=0}^{\infty} \bar{L}_n(x) \frac{t^n}{n!} , \tag{A-4.16}$$

where the expansion coefficients $\bar{L}_n(x)$ are equal to

$$\bar{L}_n(x) = \left[\frac{\partial^n \Psi(x, t)}{\partial t^n} \right]_{t=0}$$

or

$$\bar{L}_n(x) = e^x \frac{d^n}{dx^n} (x^n e^{-x}) . \tag{A-4.17}$$

From this we see that $\bar{L}_n(x)$ is a polynomial of n th degree.

Now we prove that

$$\bar{L}_n(x) \equiv L_n(x) .$$

To this end we show:

(a) The polynomial $L_n(x)$ satisfies the Tschebyscheff-Laguerre differential equation.

(b) The coefficient of x^n is equal to $a_n = (-1)^n$.

Property (b) follows easily from formula (A-4.17). Property (a) we shall prove as follows:

We set

$$\bar{L}_n(x) = e^x \frac{d^n z}{dx^n}$$

with

$$z = x^n e^{-x} .$$

Then

$$\frac{dz}{dx} = -z + \frac{nz}{x} ,$$

so that we obtain the equation

$$xz' + (x - n)z = 0$$

for z.

By $(n + 1)$ times differentiation, we obtain the identity

$$xz^{(n-2)} + (x + 1)z^{(n-1)} + (n + 1)z^{(n)} = 0 ,$$

that is, the function

$$u = \frac{d^n z}{dx^n}$$

satisfies the equation

$$xu'' + (x + 1)u' + (n + 1)u = 0 . \tag{A-4.18}$$

Now we calculate the first two derivatives of $\bar{L}_n$:

$$\bar{L}_n' = e^x(u + u') , \tag{A-4.19}$$

and

$$\bar{L}_n'' = e^x(u'' + 2u' + u) . \tag{A-4.20}$$

Using Eqs. (A-4.19), (A-4.20) and (A-4.21) we then find

$$x\bar{L}_n'' + (1 - x)\bar{L}_n' = e^x[xu'' + (1 + x)u' + u] = -n\bar{L}_n ,$$

that is, $\bar{L}_n(x)$ satisfies the differential equation

$$x\bar{L}_n'' + (1 - x)\bar{L}_n' + n\bar{L}_n = 0 .$$

With this, however, we prove that $\bar{L}_n(x) \equiv L_n(x)$.

Consequently, for the Tschebyscheff-Laguerre polynomial we obtain the differential formula

$$L_n(x) = e^x \frac{d^n}{dx^n}(e^{-x}x^n) . \tag{A-4.21}$$

Using Eq. (A-4.12) for $L_n(x)$, we can easily see that the polynomials $L_m(x)$ and $L_n(x)$ are orthogonal to each other with weight e^{-x} for $m \neq n$. We shall prove the orthogonality starting from the differential formula, since with this the norm of $L_n(x)$ is also obtained.

First, we consider the integral

$$J = \int_0^\infty L_m(x)L_n(x)e^{-x}\,dx = \int_0^\infty L_m(x)\frac{d^n}{dx^n}(x^n e^{-x})\,dx .$$

Let $m \leqq n$. By m partial integrations, we obtain

$$J = (-1)^m \int_0^\infty \frac{d^m L_m}{dx^m}\frac{d^{n-m}}{dx^{n-m}}(x^n e^{-x})\,dx ,$$

since the term which contains the factor of the form $x^k e^{-x} (k > 0)$, vanishes. If $m < n$, then by repeated partial integration and because $(d^{m+1}/dx^{m+1})L_m = 0$, we find that $J = 0$, that is, we obtain

$$\int_0^\infty L_m(x)L_n(x)e^{-x}\,dx = 0 .$$

However, if $m = n$, then we obtain

$$J = (-1)^n \int_0^\infty (-1)^n n! x^n e^{-x}\, dx = n! \Gamma(n+1) = (n!)^2 .$$

Thus we also obtain

$$\int_0^\infty L_m(x) L_n(x) e^{-x}\, dx = \begin{cases} 0 & \text{for } m \neq n , \\ (n!)^2 & \text{for } m = n , \end{cases} \tag{A-4.22}$$

In other words, the Tschebyscheff-Laguerre polynomial $L_n(x)$ forms an orthogonal system of eigenfunctions of the differential equation (A-4.12) (with weight e^{-x}) which corresponds to the eigenvalue $\lambda = n$.

The normalized polynomial has the form

$$l_n(x) = \frac{L_n(x)}{n!} = \frac{1}{n!} e^x \frac{d^n}{dx^n}(x^n e^{-x}) . \tag{A-4.23}$$

To the Tschebyscheff-Laguerre polynomials correspond the orthogonal normalized functions

$$\psi_n = \frac{e^{-x/2}}{n!} L_n(x) , \tag{A-4.24}$$

which satisfy the equation

$$(x\psi')' + \left(\frac{1}{2} - \frac{x}{4}\right)\psi + \lambda\psi = 0 \tag{A-4.25}$$

and are bounded for $x = 0$ and for $x \to \infty$ are equal to zero.

Often we also must consider the functions

$$S_n = x^{-1/2}\psi_n , \tag{A-4.26}$$

which are solutions of the self-adjoint differential equation

$$(x^2 S')' - \frac{x^2 - 2x - 1}{4} S + \lambda x S = 0 \tag{A-4.27}$$

and which for $x \to \infty$ are equal to zero. In both cases, the eigenvalues are positive integers

$$\lambda = n .$$

The orthogonality relations have the form

$$\int_0^\infty \psi_n(x)\psi_m(x)\, dx = \begin{cases} 0 & \text{for } m \neq n , \\ 1 & \text{for } m = n , \end{cases} \tag{A-4.28}$$

$$\int_0^\infty S_m(x) S_n(x) x\, dx = 0 \quad \text{for } m \neq n . \tag{A-4.29}$$

2. In the investigation of the motion of electrons in a Coulomb field, as well as in an entire series of other problems of modern physics, besides the polynomials $L_n(x)$, the generalized Tschebyscheff-Laguerre polynomials

$$y(x) = Q_n^{(s)}(x) , \qquad (s \leqq n) \tag{A-4.30}$$

occur. The polynomials (A-4.30) satisfy the differential equation

$$xy'' + (s + 1 - x)y' + \left(\lambda - \frac{s+1}{2}\right)y = 0 , \tag{A-4.31}$$

which can also be written in the self-adjoint form

$$(x^{s+1}e^{-x}y')' + x^s e^{-x}\left(\lambda - \frac{s+1}{2}\right)y = 0 . \tag{A-4.32}$$

The boundary conditions here read:

$$\begin{aligned} &y(x) \text{ is finite for } x = 0, \\ &y(x) \text{ is of order } x^n \ (n > 0 \text{ and finite}) \text{ for } x \to \infty. \end{aligned} \tag{A-4.33}$$

If we write the solution of Eq. (A-4.31) in the form

$$y = \sum_{n=0}^{\infty} a_n x^n , \tag{A-4.34}$$

then we obtain the recursion formula

$$a_{n+1} = \frac{n + (s+1)/2 - \lambda}{n(n+s)} a_n \tag{A-4.35}$$

for the coefficients. If

$$\lambda = n + \frac{s+1}{2} , \tag{A-4.36}$$

where n is a positive integer, then series (A-4.34) truncates after a finite number of terms and we obtain a polynomial as a solution.

The coefficient a_0 is arbitrarily chosen. The normalization of the solution is determined by it. If a_0 is so chosen that the coefficient of the highest power x^n is equal to $(-1)^n$

$$a_n = (-1)^n , \tag{A-4.37}$$

then we obtain the generalized Tschebyscheff-Laguerre polynomials. Obviously, these represent the unique solutions of Eq. (A-4.32) which satisfy the requirements (A-4.33) and (A-4.37).

The generating function of the polynomials $Q_n^{(s)}(x)$ is the function

$$\Psi(x, t) = \frac{e^{-xt/(1-t)}}{(1-t)^{s+1}} , \tag{A-4.38}$$

since the MacLaurin series of $\Psi(x, t)$ has the form

$$\Psi(x, t) = \sum_{n=0}^{\infty} Q_n^{(s)}(x)\frac{t^n}{n!} . \tag{A-4.39}$$

For $s = 0$, we obtain formula (A-4.16) for the Tschebyscheff-Laguerre polynomials.

For $Q_n^{(s)}(x)$, the differential formula

$$Q_n^{(s)}(x) = \frac{e^x}{x^s}\frac{d^n}{dx^n}(e^{-x}x^{n+s}) \tag{A-4.40}$$

holds.

Thus we recognize at once from

$$T_n^{(s)}(x) = \frac{e^x}{x^s}\frac{d^n}{dx^n}(e^{-x}x^{n+s}) ,$$

that $T_n^{(s)}(x)$ represents a polynomial of the n th degree, where the coefficient of x^n is equal to $(-1)^n$.

Now we show that the polynomial $T_n^{(s)}(x)$ satisfies Eq. (A-4.31).

For this purpose we introduce the function

$$z = x^{n+s}e^{-x} .$$

Obviously, in this case the relation

$$xz' + (x - n - s)z = 0$$

holds. By differentiating $(n+1)$ times with respect to x, we obtain

$$xz^{(n+2)} + (x + 1 - s)z^{(n+1)} + (n+1)z^{(n)} = 0 ,$$

that is, the function

$$u = \frac{d^n z}{dx^n}$$

satisfies the equation

$$xu'' + (x + 1 - s)u' + (n+1)u = 0 .$$

If we calculate the derivatives of $T_n^{(s)}(x)$, then by consideration of the last equation for u we obtain

$$x\frac{dT_n^{(s)}}{dx} = xT_n^{(s)} - sT_n^{(s)} + e^x x^{-s+1}u' ,$$

$$x\frac{d^2T_n^{(s)}}{dx^2} = (x - s - 1)\frac{dT_n^{(s)}}{dx} + T_n^{(s)} + e^x x^{-s}[xu'' + (x - s + 1)u']$$

$$= (x - s - 1)\frac{dT_n^{(s)}}{dx} - nT_n^{(s)} ,$$

that is, the polynomial $T_n^{(s)}(x)$ satisfies Eq. (A-4.31) for $\lambda = n + (x+1)/2$.

However, from this there follows immediately

$$T_n^{(s)}(x) \equiv Q_n^{(s)}(x) .$$

We shall now prove that the generalized polynomials with weight $x^s e^{-x}$ are orthogonal to each other in the region $0 < x < \infty$. Moreover, we shall calculate their norms.

First we consider the integral

$$J = \int_0^\infty Q_m^{(s)} Q_n^{(s)} e^{-x} x^s \, dx = \int_0^\infty Q_m^{(s)} \frac{d^n}{dx^n}(e^{-x}x^{n+s}) \, dx .$$

Let $m \leqq n$. By m partial integrations we obtain

$$J = (-1)^n \int_0^\infty \frac{d^m}{dx^m} Q_m^{(s)} \frac{d^{n-m}}{dx^{n-m}}(e^{-x}x^{n+s}) \, dx .$$

If $m < n$, then by repeated partial integrations we obtain $J = 0$ because $d^{m+1}Q_m^{(s)}/dx^{m+1} = 0$, that is, we find

$$\int_0^\infty Q_m^{(s)}(x)Q_n^{(s)}(x)e^{-x}x^s\,dx = 0 .$$

If $m = n$, then we obtain

$$J = n!\int_0^\infty e^{-x}x^{n+s}\,dx = n!\,\Gamma(n+s+1) ,$$

that is,

$$\int_0^\infty [Q_n^{(s)}(x)]^2 e^{-x}x^s\,dx = n!\,\Gamma(n+s+1) .$$

Therefore, we have

$$\int_0^\infty Q_m^{(s)}(x)Q_n^{(s)}(x)e^{-x}x^s\,dx = \begin{cases} 0 & \text{for } m \neq n , \quad (s > -1) , \\ n!\,\Gamma(n+s+1) & \text{for } m = n . \end{cases} \tag{A-4.41}$$

To the generalized Tschebyscheff-Laguerre polynomials $Q_n^{(s)}(x)$ correspond the orthonormalized functions

$$\Phi_n^{(s)}(x) = \frac{x^{s/2}e^{-x/2}Q_n^{(s)}(x)}{\sqrt{n!\,\Gamma(n+s+1)}} \tag{A-4.42}$$

$$\int_0^\infty \Phi_n^{(s)}(x)\Phi_m^{(s)}(x)\,dx = \begin{cases} 0 & \text{for } m \neq n , \\ 1 & \text{for } m = n , \end{cases}$$

which are solutions of the equation

$$\frac{d}{dx}\left(x\frac{d\Phi}{dx}\right) + \left(\lambda - \frac{x}{4} - \frac{s^2}{4x}\right)\Phi = 0 \tag{A-4.43}$$

for the eigenvalue

$$\lambda = n + \frac{s+1}{2} .$$

The boundary conditions which these functions satisfy are obvious from the above observations. Often we use the normalized polynomials

$$Q_n^{*(s)}(x) = \frac{Q_n^{(s)}(x)}{\sqrt{n!\,\Gamma(n+s+1)}} . \tag{A-44.4}$$

3. Simple problems for the Schrödinger equation[58]

1. *The Schrödinger equation.* In quantum mechanics the behavior of the particles in a potential field are described by the Schrödinger equation

$$i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2\mu}\Delta\psi + U(x, y, z, t)\psi . \tag{A-4.45}$$

[58] The problems treated here for the Schrödinger equation are considered as examples of the application of the Tschebyscheff-Hermite and Tschebyscheff-Laguerre polynomials. Our investigations make no claim to the complete clarification of questions related to the Schrödinger equation.

Here $\hbar = 1.05 \cdot 10^{-27}$ erg sec denotes Planck's constant, μ is the mass of the particles, U its potential energy in the force field, and $\psi = \psi(x, y, z, t)$ is the wave function; further, $i = \sqrt{-1}$.

If the forces are time independent, $U = U(x, y, z)$, then stationary states for the given energy state are possible, that is, there exist solutions of the form

$$\psi = \psi^0(x, y, z)e^{-(iE/\hbar)t} , \tag{A-4.46}$$

where E is the total energy of the particles. If we put this expression into Eq. (A-4.45), we arrive at the second Schrödinger equation

$$\Delta\psi^0 + \frac{2\mu}{\hbar^2}(E - U)\psi^0 = 0 , \tag{A-4.47}$$

in which E plays the role of an eigenvalue yet to be determined. In the following discussion, in place of ψ^0 we shall write ψ:

$$\Delta\psi + \frac{2\mu}{\hbar^2}(E - U)\psi = 0 . \tag{A-4.48}$$

If no force field is present, then $U = 0$, and Eq. (A-4.48) assumes the form

$$\Delta\psi + \frac{2\mu E}{\hbar^2}\psi = 0 . \tag{A-4.49}$$

We immediately recognize the similiarity of this equation with the vibration equation of classical physics

$$\Delta\psi + k^2\psi = 0 , \tag{A-4.50}$$

in which $k = \omega/c = 2\pi/\lambda$ (λ is the wave length) denotes the wave number. However, this similiarity is purely superficial and formal because of the different physical meaning of the functions occurring in Eqs. (A-4.49) and (A-4.50).

In the Schrödinger equation, the function ψ itself has no direct physical meaning, only the function $|\psi|^2$. This can be explained statistically in the following manner: The expression $|\psi|^2\,dx\,dy\,dz$ denotes the probability of finding the particle considered within the volume element $dx\,dy\,dz$ at the point x, y, z.

In this connection, the normalization of the eigenfunctions to one, which we repeatedly impose for the purpose of mathematical simplicity, now has a basic significance. The normalization condition

$$\iiint |\psi|^2\,dx\,dy\,dz = 1 \tag{A-4.51}$$

means that the particle can be found at a certain point of space and that, therefore, the probability of a particle being found anywhere in space is equal to one (a certain event).

2. *The harmonic oscillator.* The Schrödinger equation for the harmonic oscillator reads

$$\frac{h^2}{2\mu}\frac{d^2\phi}{dx^2} + (E - U)\phi = 0 .$$

Here $U = (\mu\omega_0^2/2)x$ and ω_0 is the eigenfrequency of the (cyclic) oscillator. Our problem, therefore, is to determine the stationary states. Thus we have to determine the spectrum of the eigenvalues of the energy E and the corresponding eigenfunctions ϕ from the equation

$$\phi'' + \frac{2\mu}{\hbar^2}\left(E - \frac{\mu\omega_0^2}{2}x^2\right)\phi = 0 \tag{A-4.52}$$

with the auxiliary condition

$$\int_{-\infty}^{\infty} |\phi|^2\, dx = 1 . \tag{A-4.53}$$

With the relations

$$\lambda = \frac{2E}{\hbar\omega_0} ,$$

$$x_0 = \sqrt{\frac{\hbar}{\mu\omega_0}} , \tag{A-4.54}$$

$$\xi = \frac{x}{x_0} ,$$

we then obtain for the function $\phi = \phi(\xi)$, after a few simple transformations, the equation

$$\frac{d^2\phi}{d\xi^2} + (\lambda - \xi^2)\phi = 0 \tag{A-4.55}$$

with the auxiliary normalization condition

$$\int_{-\infty}^{\infty} |\phi|^2\, d\xi = \frac{1}{x_0} . \tag{A-4.56}$$

On the basis of the observation made at the end of Section A-4.1, the functions

$$\phi_n(\xi) = \frac{1}{\sqrt{x_0}}\,\frac{e^{-\xi^2/2}H_n(\xi)}{\sqrt{2^n n!}\,\sqrt{\pi}}$$

represent the solutions of these problems. They correspond to the eigenvalue

$$\lambda_n = 2n + 1 .$$

In the original coordinates, we have

$$\phi_n(x) = \frac{1}{\sqrt{x_0}}\,\frac{e^{-(1/2)(x/x_0)^2}H_n(x/x_0)}{\sqrt{2^n n!}\,\sqrt{\pi}} , \tag{A-4.57}$$

$$E_n = \hbar\omega_0\left(n + \frac{1}{2}\right) , \qquad (n = 0, 1, 2, \ldots) . \tag{A-4.58}$$

In classical mechanics, the energy of the oscillator

$$E = \frac{p_x^2}{2\mu} + \frac{\mu\omega_0^2}{2}x^2$$

(p_x denotes the momentum) can assume a continuous set of values. From the standpoint of quantum mechanics, on the other hand, as formula (A-4.58) shows, the oscillator can have only a discrete set of energy values E_n. The number n which gives the number of quantum levels is called the principal quantum number. In the lowest quantum state for $n = 0$, the energy of the oscillator remains different from zero; in fact, it is equal to

$$E_0 = \frac{1}{2}\hbar\omega .$$

3. *The eigenvalue problem of rotators.* In the following discussion, we shall investigate the eigenvalues of the energy of a rotator about a spatially fixed axis, that is, particles which rotate at one and the same distance about a fixed center.

The potential energy U of the rotators for all positions of the particles has a fixed value and can be set equal to zero, $U = 0$.

In a polar coordinate system (r, θ, φ) whose origin lies at the fixed center, the Schrödinger equation for the rotator

$$\Delta\psi + \frac{2\mu}{\hbar^2}E\psi = 0$$

can be written in the form

$$\frac{1}{r^2 \sin\theta}\left(\sin\theta\frac{\partial\psi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\psi}{\partial\varphi^2} + \frac{2\mu}{\hbar^2}E\psi = 0 . \tag{A-4.59}$$

Here the condition

$$\frac{\partial\psi}{\partial r} = 0$$

is used.

If, in place of the mass μ, the moment of inertia

$$I = \mu r^2$$

is introduced, we obtain

$$\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2\psi}{\partial\varphi^2} + \lambda\psi = 0$$

or

$$\Delta_{\theta,\varphi}\psi + \lambda\psi = 0 , \tag{A-4.60}$$

where

$$\lambda = \frac{2I}{\hbar^2}E . \tag{A-4.61}$$

Thus we arrive at the eigenvalue problem for the equation

$$\Delta_{\theta,\varphi}\psi + \lambda\psi = 0 . \tag{A-4.60}$$

In addition, it is required that the eigenfunctions for $\theta = 0$ and $\theta = \pi$ be bounded and the normalization condition

$$\int_0^{\pi}\int_0^{2\pi} |\psi|^2 \sin\theta \, d\theta \, d\varphi = 1 \tag{A-4.62}$$

be satisfied. As we know, the normalized spherical functions

$$\psi_{lm}(\theta, \varphi) = \sqrt{\frac{(2l+1)(l-m)!}{2\varepsilon_m \pi (l+m)!}} \, Y_l^{(m)}(\theta, \varphi) \,, \qquad \left(\varepsilon_m = \begin{cases} 2 & \text{for } m = 0 \\ 1 & \text{for } m \neq 0 \end{cases}\right), \tag{A-4.63}$$

$$Y_l^{(m)}(\theta, \varphi) = P_l^{(m)}(\cos\theta) \begin{cases} \cos m\varphi \,, \\ \sin m\varphi \,, \end{cases} \qquad (m = 0, 1, \ldots, l)$$

are the solutions of this problem. They correspond to the eigenvalue

$$\lambda = l(l+1) \,. \tag{A-4.64}$$

If we replace λ according to Eq. (A-4.61), then for the energy value of the rotators we obtain

$$E_{lm} = l(l+1)\frac{\hbar^2}{2I} \,. \tag{A-4.65}$$

4. *Motion of electrons in a Coulomb field.* One of the simplest problems of atomic theory is the problem of the motion of electrons in the Coulomb field of a nucleus. This problem has extensive practical significance since its solution leads not only to the theory of the spectrum of hydrogen, but also to an approximate theory of the spectra of atoms with one valence electron (hydrogenlike atoms). Thus, for example, the problem pertains to the sodium atom.

In the hydrogen atom we have a single electron in the Coulomb field of the nucleus (the protons), so that the potential energy $U(x, y, z)$ is equal to

$$U = -\frac{e^2}{r} \,, \tag{A-4.66}$$

where r is the distance of the electrons from the nucleus, $-e$ is the charge of the electron, and $+e$ the charge of the nucleus.

The Schrödinger equation reads

$$\Delta\psi + \frac{2\mu}{\hbar^2}\left(E + \frac{e^2}{r}\right)\psi = 0 \,. \tag{A-4.67}$$

Our problem is to determine those energy values E for which Eq. (A-4.67) admits solutions that are continuous in the entire space and that satisfy the normalization condition

$$\int\int_{-\infty}^{\infty}\int |\psi(x, y, z)|^2 \, dx \, dy \, dz = 1 \,. \tag{A-4.68}$$

First we introduce a spatial polar coordinate system whose origin is in the nucleus. The nucleus itself we regard as being at rest. In this system, the Schrödinger equation assumes the form

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial\psi}{\partial r}\right) + \frac{1}{r^2}\Delta_{\theta,\varphi}\psi + \frac{2\mu}{\hbar^2}\left(E + \frac{e^2}{r}\right)\psi = 0 \,. \tag{A-4.69}$$

For the solution, we make the Ansatz

$$\psi(r, \theta, \varphi) = \chi(r) Y_l^{|m|}(\theta, \varphi) . \tag{A-4.70}$$

As is known, the spherical functions $Y_l^{|m|}(\theta, \varphi)$ satisfy the differential equation

$$\Delta_{\theta,\varphi} Y_l^{|m|}(\theta, \varphi) + l(l+1) Y_l^{|m|}(\theta, \varphi) = 0 ,$$

and we obtain

$$\frac{d^2\chi}{dr^2} + \frac{2}{r}\frac{d\chi}{dr} + \left[\frac{2\mu}{\hbar^2}\left(E + \frac{e^2}{r}\right) - \frac{l(l+1)}{r^2}\right]\chi = 0 . \tag{A-4.71}$$

Henceforth, the quantity

$$a = \frac{\hbar^2}{\mu e^2} = 0.529 \cdot 10^{-8}\,\text{cm}$$

will be used as the unit of length and the quantity

$$E_0 = \frac{\mu e^4}{\hbar^2} = \frac{e^2}{a}$$

as the unit of energy. Furthermore, we set

$$\rho = \frac{r}{a} , \qquad \varepsilon = \frac{E}{E_0} (\varepsilon < 0) . \tag{A-4.72}$$

Then we can also write

$$\frac{d^2\chi}{d\rho^2} + \frac{2}{\rho}\frac{d\chi}{d\rho} + \left(2\varepsilon + \frac{2}{\rho} - \frac{l(l+1)}{\rho^2}\right)\chi = 0 \tag{A-4.73}$$

for Eq. (A-4.71). With the help of the substitution

$$\chi = \frac{1}{\sqrt{\rho}} y , \tag{A-4.74}$$

we bring Eq. (A-4.73) to the form

$$\frac{d^2y}{d\rho^2} + \frac{1}{\rho}\frac{dy}{d\rho} + \left(2\varepsilon + \frac{2}{\rho} - \frac{s^2}{4\rho^2}\right)y = 0 , \tag{A-4.75}$$

where

$$s = 2l + 1 .$$

If we now introduce

$$x = \rho\sqrt{-8\varepsilon} \tag{A-4.76}$$

as the independent variable, then in place of Eq. (A-4.75) we obtain the equation

$$xy'' + y' - \left(\frac{x}{4} + \frac{s^2}{4x}\right)y + \lambda y = 0 \tag{A-4.77}$$

or

$$\frac{d}{dx}\left(x\frac{dy}{dx}\right) - \left(\frac{x}{4} + \frac{s^2}{4x}\right)y + \lambda y = 0 , \tag{A-4.77'}$$

where

$$\lambda = \frac{1}{\sqrt{-2\varepsilon}} . \tag{A-4.78}$$

This has already been investigated in Section A-4.2 (see Eq. (A-2.48)).

Its eigenvalues were found to be

$$\lambda = n_r + \frac{s+1}{2} ,$$

whereas the eigenfunctions can be expressed in terms of the generalized Tschebyscheff-Laguerre polynomials $Q_{n_r}^{*(s)}$:

$$y_{n_r}(x) = x^{s/2} e^{-x/2} Q_{n_r}^{*(s)}(x) . \tag{A-4.79}$$

The polynomials $Q_{n_r}^{*(s)}$ are defined by formula (A-4.43).

Since

$$s = 2l + 1 ,$$

we find

$$\lambda = n_r + l + 1 = n , \qquad (n = 1, 2, \ldots) . \tag{A-4.80}$$

The integer n is called the principal quantum number, while n_r is called the radial and l the azimuthal quantum number (secondary quantum number).

If we replace λ according to Eqs. (A-4.78) and (A-4.72), we obtain the energy values

$$E_n = -\frac{\mu e^4}{2\hbar^2 n^2} . \tag{A-4.81}$$

Thus these depend only on the principal quantum number n.

Now we set E equal to the energy of a quantum $\hbar\omega = h\nu$, $E = -h\nu$. Here $\nu = \omega/2\pi$ denotes the frequency. Then

$$\nu = \frac{\mu e^4}{2\hbar^2 n^2 h} = \frac{R}{n^2} , \tag{A-4.82}$$

where $R = (\mu e^4)/(2\hbar^2 h)$ is the so-called Rydberg constant.

We shall now examine the frequencies of the spectral lines. The observed frequency of a spectral line corresponds to the transition of a state with energy E_n to a state with energy E_{n_1}.

The frequency ν_{nn_1} of a quantum which is emitted by such a transition is equal to

$$\nu_{nn_1} = R\left(\frac{1}{n_1^2} - \frac{1}{n^2}\right) . \tag{A-4.83}$$

If we set $n_1 = 1$ and let n successively take the values $n = 2, 3, \ldots$, then we obtain a series of lines which together form the so-called Lyman series

$$\nu_{nn_1} = R\left(1 - \frac{1}{n^2}\right) .$$

For the values $n_1 = 2$, $n = 3, 4, \ldots$, we obtain the Balmer series

$$\nu_{nn_1} = R\left(\frac{1}{2^2} - \frac{1}{n^2}\right),$$

and for $n_1 = 3$, $n = 4, 5, \ldots$, the Paschen series

$$\nu_{nn_1} = R\left(\frac{1}{3^2} - \frac{1}{n^2}\right).$$

We turn now to the determination of the eigenfunctions of the hydrogen atom. According to formula (A-4.70), it is sufficient to determine the radial functions $\chi(\rho)$.

By applying formulas (A-4.79), (A-4.76), (A-4.74), (A-4.78), and (A-4.80), we can write

$$\chi_{nl}(\rho) = A_n\left(\frac{2\rho}{n}\right)^l e^{-\rho/n} Q_{n-l-1}^{*(2l+1)}\left(\frac{2\rho}{n}\right). \tag{A-4.84}$$

Here A_n denotes the normalization factors to be determined, which can be calculated from the condition

$$\int_0^\infty \rho^2 \chi_{nl}^2(\rho)\, d\rho = 1\,. \tag{A-4.85}$$

From the calculation of A_n, we obtain for the normalized radial functions the expression

$$\chi_{nl}(\rho) = \frac{2}{n^2}\left(\frac{2\rho}{n}\right)^l e^{-\rho/n} Q_{n-l-1}^{*(2l+1)}\left(\frac{2\rho}{n}\right). \tag{A-4.86}$$

From formula (A-4.70) and functions (A-4.63), the normalized eigenfunctions then have the form

$$\psi_{mnl} = \sqrt{\frac{(2l+1)(l-m)!}{2\varepsilon_m \pi (l+m)!}}\, Y_l^{(m)}(\vartheta, \varphi)\chi_{nl}(\rho)\,,$$

where the $\chi_{nl}(\rho)$ are to be taken according to expression (A-4.86).

The number $m (m = 0, \pm 1, \pm 2, \ldots, \pm l)$ is called the magnetic quantum number.

Since n_r is nonnegative ($n_r = 0, 1, 2, \ldots$), the quantum number l cannot be larger than $n - 1$ ($l = 0, 1, 2, \ldots, n - 1$) for a prescribed principal quantum number n because

$$n = n_r + l + 1\,.$$

Therefore, for a fixed value of the principal quantum number n, l can assume the n values $l = 0, 1, \ldots, n - 1$. However, here each value of l corresponds exactly to the $2l + 1$ value of m. It also follows that to a given energy value E_n, that is, to a given value n, belong

$$\sum_{l=0}^{n-1} (2l + 1) = 1 + 3 + 5 + \cdots + (2n - 1) = n^2$$

distinct eigenfunctions. The multiplicity of the degeneracy of each energy level is thus equal to n^2.

The discrete spectrum of the negative eigenvalues of the energy E_n consists of infinitely many numbers whose limit point is the number zero.

The problem considered above for the Schrödinger equation can be distinguished further by the existence of a continuous spectrum of positive eigenvalues [each positive number E is an eigenvalue of Eq. (A-4.6)]. In this case, the electron is not bound to the nucleus, nevertheless it exists in a field (ionized hydrogen atoms). For the proof of the existence of this complicated spectrum, we refer the reader to the special literature.†

† Editors note: G. Hellwig, *Differential Operators of Mathematical Physics*, Springer-Verlag, Berlin, (1964); R. Courant and D. Hilbert, *Methods of Mathematical Physics*, Vol. I. Interscience, 1953.

INDEX

Absorption
  first type, 447
  second type, 447
Approximation theorem, Weierstrass, 575

Balmer series, 617
Bessel differential equation, 472, 525, 532-3
Biot-Savart law, 412
Boundry value problem
  of Bessel's differential equation, 542, 544
  exterior first, 599
  solution of, 530

Cavity resonator, 507
Chain reaction, 435
Chladni's sound figure, 398
Completeness relation, 575
Condition, Leontowitch, 514
Conditions, Sommerfeld, 483
Cone, characteristics, 389
Constant, Rydberg, 616
Constants, Lamé, 409
Cylindrical functions, 545

d'Alembert's formula, 385
Descending method, 383-385
Differential equation
  Bessel, 472, 525, 532, 533
  of cylindrical functions, 532
  Euler, 585
  Laplace's, 585
  Legendre, 525
  solution of Laplace's, 590
  Tschebyscheff-Hermite, 526
  Tschebyscheff-Laguerre, 526
Differential formula of the Tschebyscheff-Laguerre polynomial, 606
Diffusion equation, 422
Dimension, critical, 435
Displacement current, 411

Endo-vibrator, 507
Equation
  Parseval's, 575
  Schrödinger, 610, 614
Equations
  of electrostatics, 412
  Maxwell's, 410, 498, 507, 515
  of magnetostatics, 412
Euler's differential equation, 585
Euler's formulas, 559
Expansion theorem, 395, 531, 544

Faraday's induction law, 412
Fock, V. A., 519, 523
Formula
  d'Alembert, 385
  Poisson's, 383, 442
  of Rodrigues, 577
  Sommerfeld's, 563
Formulas, Euler's, 559
Fourier-Bessel integral, 560
Fourier development, 443-4
Fourier's integral, 559
Fredholm's integral equation of the second type, 476
Function
  Bessel, 536, 555, 559, 566
  Green's, 424, 440, 442-3, 591
  Hankel's, 545, 559, 568
  Neumann's, 545, 559

Gamma function, 534
Green's formula, 468, 589
Green's function, 424, 440, 442-3, 591

Hankel's function, 545, 559, 568
Heat conduction equation
  fundamental solution, 438
  solution of, 442, 426, 440, 458
Hertz vector, 418
Hooke's law, 408
Huyghen's principle, 386

Ignatowsky, W. S., 483
Induction flux, 411, 454
Induction law, Faraday's, 410
Integral
  Fourier-Bessel, 560
  Fourier's, 559
Integral equation

of the second type, Fredholm, 476
Volterra, 448

Kirchhoff's wave formulas, 391-394

Lamé constants, 409
Landau symbols, 480
Laplace differential equation, 585
Laplace differential equation, solution of, 590
Law
Biot-Savart's, 412
Hooke's, 408
Legendre differential equation, 525
Legendre polynomial, 572
associated, 580
generating function of, 572
Leontowitch, M. A., 507
Leontowitch, condition of, 514
Limiting absorption, 476-7
Limiting amplitude, 478
Lyman series, 616

Material equations, 410
Maxwell equations, 410, 498, 507, 515
Mean value method, 381

Neumann's function, 545, 559
Nodal lines, 398
Nodal surfaces, 398
Norm of an eigenfunction, 397
Normal stresses, 408

Parseval equation, 575
Pascal series, 617
Poisson formula, 383, 442
Polarization potential, 504
Pol, B. Van der, 523
Polynomial
associated Legendre, 580
generalized Tschebyscheff-Laguerre, 607
generating function of Legendre, 572
harmonic, 584
Legendre, 572
Tschebyscheff, 558
Tschebyscheff-Hermite, 601
Tschebyscheff-Laguerre, 604
Potential of a
double layer, 474
single layer, 475
Poynting's vector, 421
Principal quantum number, 616
Principle, Huyghen's, 386, 387

Quantum number, magnetic, 617

Radiation conditions, 483
Rayleigh (J. W. Strutt), 498
Reciprocity, principle of, 472
Recursion formula for the
Bessel functions, 538
Legendre functions, 573
Reflection method, 388
Rodrigues, formula of, 577
Rydberg constant, 616

Schrödinger equation, 610, 614
Series
Balmer, 617
Lyman, 616
Pascal, 617
Shear
modulus, 409
stress, 408
Skin effect, 515
Sobolev, S. L., 391
Sommerfeld, A., 519, 523, 563
Sommerfeld
conditions, 483
formula, 563
Sound figures, Chladni, 398
Spherical functions, 585
general, 585
rational, 585
tesseral, 587
zonal, 587
Spherical surface functions, 585
Spherical waves, 480
Strain tensor, 409
Stress tensor, 408
Symbols, Landau, 480
System
closed, 575
complete, 575, 583

Tangential stresses, 408
Tschebyscheff-Hermite
differential equation, 526
polynomial, 601
Tschebyscheff-Laguerre
differential equation, 526
polynomial, 604
Tschebyscheff-Laguerre polynomial,
generalized, 607
Tschebyscheff polynomial, 558

Uniqueness theorem, 476

Vector
field of an oscillator, 418
Hertz, 418
Poynting, 421
Vekua, I. N., 482

Vibration equation, 465
  solution of, 479, 546
Volume potential, 474
Volterra integral equation, 448

Wave
  planar, 480, 489
  standing, 398
  zone of oscillator, 421
Wave equation, 393
  solution of, 387, 392
Wave formula, Kirchhoff, 391-394
Weierstrass approximation theorem, 575
Weil, A., 523